Tutorial session
(14/09/24)

Leibniz Rule

Theorem 2. *Suppose that f and $\frac{\partial f}{\partial x}$ are continuous in the rectangle*

$$R = \{(x, t) : a \le x \le b, c \le t \le d\}$$

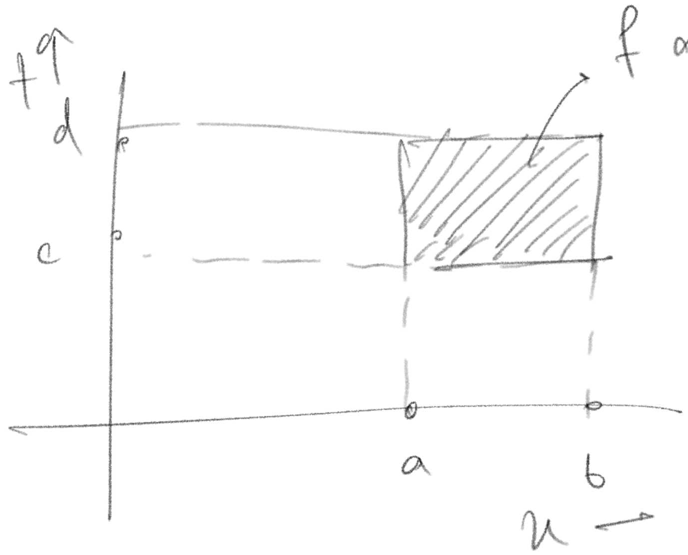
and suppose that $u_0(x)$ and $u_1(x)$ are continuously differentiable for $a \le x \le b$ with the range of $u_0(x)$ and $u_1(x)$ in (c, d). If ψ is given by

$$\psi(x) = \int_{u_0(x)}^{u_1(x)} f(x, t)\, dt \tag{12}$$

then

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{\partial}{\partial x} \int_{u_0(x)}^{u_1(x)} f(x, t)\, dt \\ &= f(x, u_1(x)) \frac{du_1(x)}{dx} - f(x, u_0(x)) \frac{du_0(x)}{dx} + \int_{u_0(x)}^{u_1(x)} \frac{\partial f(x,t)}{\partial x} dt \end{aligned} \tag{13}$$

If one of the bounds of integration does not depend on x, then the term involving its derivative will be zero.



f and $\frac{\partial f}{\partial x}$, $\frac{\partial u_1(x)}{\partial x}$, $\frac{\partial u_2(x)}{\partial x}$ are all continuous

Q.1) If X_1, X_2 are IID with
$X_i \sim \mathcal{N}(0,1)$ for each $i \in \{1,2\}$.
Then, pdf of $X_1 + X_2$?

Sol. 1 we know that,
$X_1 \in \mathbb{R}$, $X_2 \in \mathbb{R}$

then, Let $Z = X_1 + X_2$,
Now, using convolution of f_{X_1} and f_{X_2},
we can get,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1, X_2}(x_1, z - x_1)\, dx_1$$

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1) \cdot f_{X_2}(z - x_1)\, dx_1$$

$\{ \because X_1 \perp\!\!\!\perp X_2 \}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x_1)^2}{2}}\, dx_1$$

$$f_2(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(2x_1^2 - 2x_1 z + z^2)} \, dx_1$$

Now, completing the square (in exponential),

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(2x_1^2 - 2x_1 z + \left(\frac{z}{\sqrt{2}}\right)^2 - \left(\frac{z}{\sqrt{2}}\right)^2 + z^2\right)} \, dx_1$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(\sqrt{2}\, x_1 - \frac{z}{\sqrt{2}}\right)^2 + \frac{z^2}{2}\right)} \, dx_1$$

$$= \frac{1}{2\pi} \cdot e^{-z^2/4} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\sqrt{2}\, x_1 - \frac{z}{\sqrt{2}}\right)^2} \, dx_1$$

Let $\sqrt{2}\, x_1 - \frac{z}{\sqrt{2}} = t$

then,

$$\sqrt{2}\, dx_1 = dt$$

$$\Rightarrow f_2(z) = \frac{1}{2\pi\sqrt{2}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \, dt$$

and, we know that,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

$$\therefore \quad f_Z(z) = \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{z^2}{2(2)}}$$

thus, $\boxed{f_Z(z) \sim \mathcal{N}(0, 2)}$

{in general} if X_1, X_2 are Independent

with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$

for $i \in \{1, 2\}$

we have,

$$f_{X_1+X_2+\dots X_n}(x_1 + x_2 + \dots x_n) = \mathcal{N}\left(\mu_1 + \mu_2 + \dots \mu_n,\ \sigma_1^2 + \sigma_2^2 + \dots \sigma_n^2\right)$$

Q.2) If X_1, X_2 are I.I.D with $X_i \sim Exp(\lambda)$, $i \in \{1, 2\}$.
find the pdf of $X_1 + X_2$?

Sol ② Now, if $X \sim Exp(\lambda)$

then,
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} , & x \geq 0 \\ 0 , & o/w \end{cases}$$

Now, let $Z = X_1 + X_2$,

Then,
$$F_Z(z) = Pr(Z \leq z)$$
$$= Pr(X_1 + X_2 \leq z)$$

Now, let $A = \{\{X_1 \geq 0\} \cap \{X_2 \geq 0\}\}$
$$= Pr((X_1 + X_2 \leq z) \cap (A \cup \bar{A}))$$
$$(\because A \cup \bar{A} = \Omega)$$

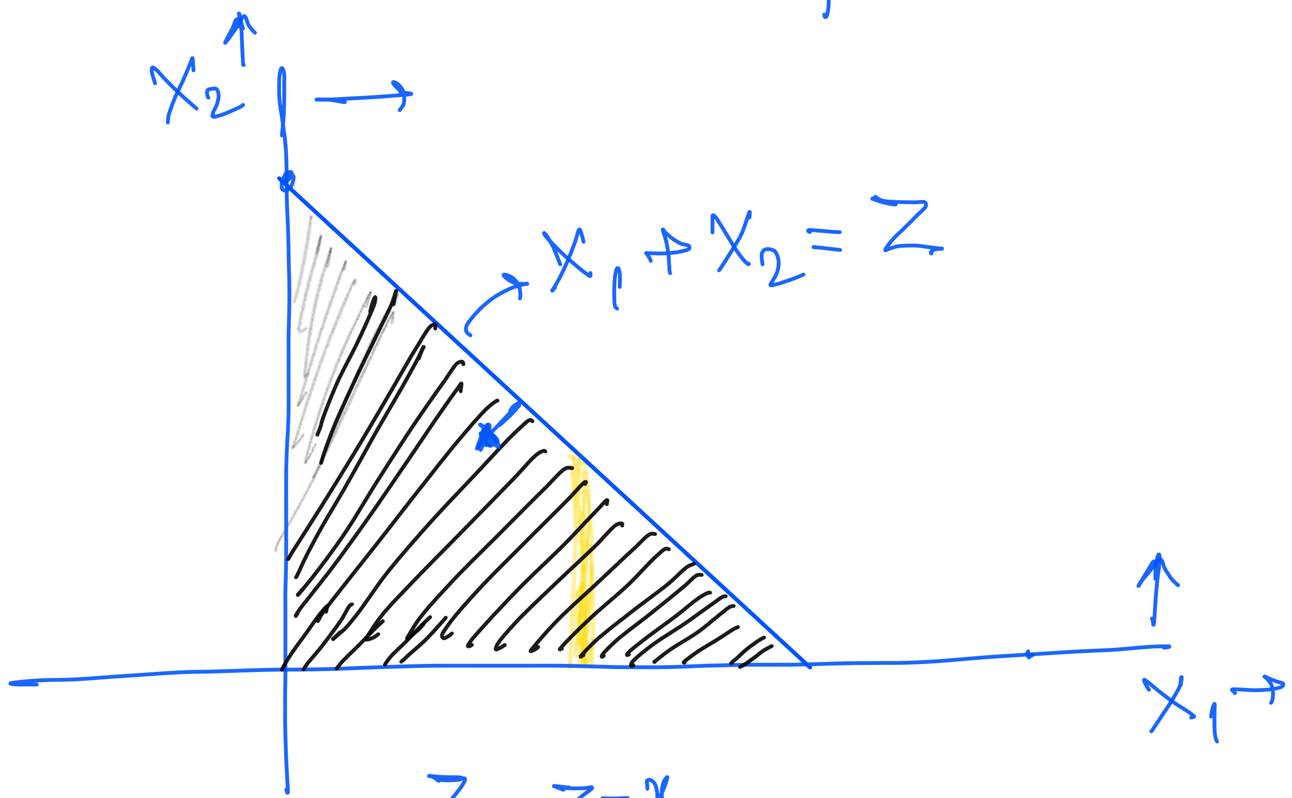
$$F_Z(z) = \Pr\left((X_1 + X_2 \le z) \cap A\right)$$

$$+ \underbrace{\Pr\left((X_1 + X_2 \le z) \cap \bar{A}\right)}_{0}$$

($\because$ X_1 and X_2 are non-negative)

$$= \Pr\left((X_1 + X_2 \le z),\ X_1 \ge 0,\ X_2 \ge 0\right)$$

Now, graphically (for finding region of integration)



So,

$$F_Z(z) = \int_0^{z} \int_0^{z - x_1} f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$$

Now, differentiating both sides wrt z,

(for getting $f_Z(z) = \dfrac{d}{dz} F_Z(z)$)

Using Leibniz rule, we get.

$$f_Z(z) = 1 \cdot \left(\int_0^{z-x_1} f_{X_1,X_2}(x_1, x_2)\, dx_2 \right) \Bigg|_{x_1 = z} + \int_0^z \left(1 \cdot f_{X_1,X_2}(x_1, z - x_1) + 0 \right) dx_1$$

$$= \int_0^{\overset{0 \leftarrow}{z-z}} f_{X_1,X_2}(x_1, x_2)\, dx_2 + \int_0^z f_{X_1,X_2}(x_1, z - x_1)\, dx_1$$

$$= \int_0^z f_{X_1}(x_1) \cdot f_{X_2}(z - x_1)\, dx_1 \quad (\because X_1 \perp\!\!\!\perp X_2)$$

$$= \int_0^z \lambda . e^{-\lambda x_1} . \lambda e^{-\lambda (z - x_1)} \, dx_1$$

$$= \lambda^2 \int_0^z e^{-\lambda z} \, dx_1$$

$$\Rightarrow \boxed{f_Z(z) = \lambda^2 z . e^{-\lambda z}}$$

In general, if $X_1, X_2 \dots X_n$ are I.I.D with $X_i \sim Exp(\lambda)$ $\forall i \in \{1, 2 \dots n\}$

Then, $$f_{X_1 + X_2 + \dots X_n}((x_1 + x_2 + \dots x_n) = x) = \frac{\lambda^n . x^{n-1} . e^{-\lambda x}}{(k-1)!}$$

Q. 3 If $X \sim N(0,1)$ and $A = \{X > 0\}$ then $f_{X|A}(x) = ?$

Sol^n: In general, let $A = \{X \in [a,b]\}$

Then, (Start by calculating CDF)

$$F_{X|A}(x) = Pr(X \leq x \mid A)$$

$$= \frac{Pr(X \leq x, A)}{Pr(A)}$$

$$= \frac{Pr(X \leq x, a \leq X \leq b)}{Pr(A)}$$

Case (i) $x < a$,

then, $Pr(X \leq x, a \leq X \leq b) = 0$

$$\Rightarrow F_{X|A}(x) = 0.$$

Case (ii) $a \leq x \leq b$, then

$$Pr(X \leq x, a \leq X \leq b) = Pr(a \leq X \leq x)$$

$$= F_X(x) - F_X(a)$$

So, $F_{X|A}(a) = \dfrac{F_X(x) - F_X(a)}{Pr(A)}$

Case (iii): $x > b$

then, $Pr(X \le x,\ a \le X \le b)$
$= Pr(a \le X \le b)$
$= Pr(A)$

$\therefore F_{X|A}(x) = 1$

In general, combing all cases, we have

$$\therefore F_{X|A}(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{F_X(x) - F_X(a)}{Pr(A)} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

$$\text{So, } f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{Pr(A)} & ;\ \text{if } a \le x \le b \\ 0 & ;\ \text{otherwise} \end{cases}$$

Thus, Now, Acc to question,

$$f_X(x) = \mathcal{N}(0,1)$$

$$A = \{X > 0\}$$

we have,

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\Pr(A)} & ; \quad x \in A \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

where, $f_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$

and $\Pr(A) = \Pr(X > 0) = 1/2$.

as, X is symmetric around 0. $(P(X>0) = P(X<0))$

i.e. $P(X>0) + P(X=0) + P(X<0) = 1$

0

and $2P(X>0) = 1$

$\Rightarrow P(X>0) = 1/2$.

Q. 4 If X and Y be continuous R.V.s. $(Y \neq 0)$

then, Let $Z = \frac{X}{Y}$ be R.V. Then find pdf of Z?

Sol^n:

$$F_Z(z) = \Pr(Z \leq z)$$

$$= \Pr\left(\frac{X}{Y} \leq z\right)$$

Let $A = \{Y > 0\}$ $\quad A \cup \bar{A} = \Omega$

then, $\bar{A} = \{Y < 0\}$ and $A \cap \bar{A} = \phi$

$$\therefore F_Z(z) = \Pr\left(\frac{X}{Y} \leq z \cap (A \cup \bar{A})\right)$$

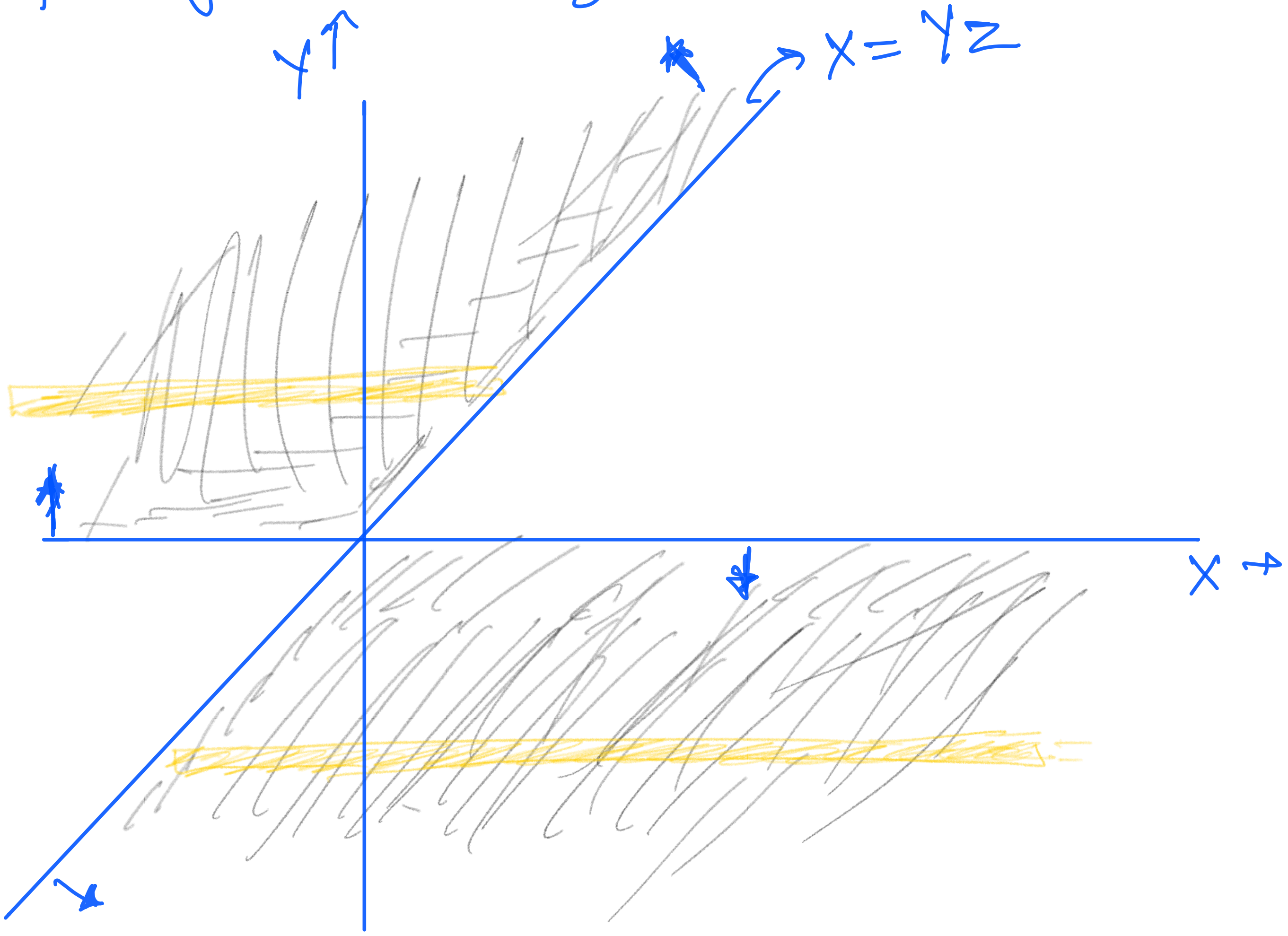
$$= \Pr\left(\left(\frac{X}{Y} \leq z\right) \cap (Y > 0)\right)$$

$$+ \Pr\left(\left(\frac{X}{Y} \leq z\right) \cap (Y < 0)\right)$$

$$= \Pr(X \le YZ,\ Y > 0)$$

$$+ \Pr(X \geq YZ,\ Y < 0)$$

Now, sketching the region of integration for joint pdf of X and Y.



Let Region I : $\{X \le YZ,\ Y \geq 0\}$

and Region II : $\{X \geq YZ,\ Y < 0\}$

$$F_Z(z) = \int_0^{\infty}\int_{-\infty}^{yz} f_{XY}(x,y)\,dx\,dy$$

$$+ \int_{-\infty}^{0}\int_{yz}^{\infty} f_{XY}(x,y)\,dx\,dy$$

Then, we know $f_Z(z) = \dfrac{d}{dz} F_Z(z)$

$\therefore$

$$f_Z(z) = \int_0^{\infty} y \cdot f_{XY}(yz, y)\,dy$$

$$+ \int_{-\infty}^{0} -y\, f_{XY}(yz, y)\,dy$$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_{XY}(yz, y)\,dy}$$

Q.5)

Let X, Y be IID Standard Normal Gaussian R.V. then,

find pdf of $\frac{X}{Y}$?

Sol.5)

Let $Z = \frac{X}{Y}$, we know that

$$f_Z(z) = \int_{-\infty}^{\infty} |y| \, f_{XY}(yz, y) \, dy$$

$$= \int_{-\infty}^{\infty} |y| \cdot f_X(yz) \cdot f_Y(y) \, dy$$

$(\because X \perp\!\!\!\perp Y)$

$$= \int_{-\infty}^{\infty} |y| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(yz)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y| \cdot e^{-\frac{(z^2+1)y^2}{2}} \, dy.$$

$\because |y| e^{-\frac{(z^2+1)y^2}{2}}$ is even function $(f(-x) = f(x))$

$$\therefore = \frac{2}{2\pi} \int_0^{\infty} y \cdot e^{-\frac{(z^2+1)y^2}{2}} dy$$

Let $\frac{(z^2+1)y^2}{2} = t$

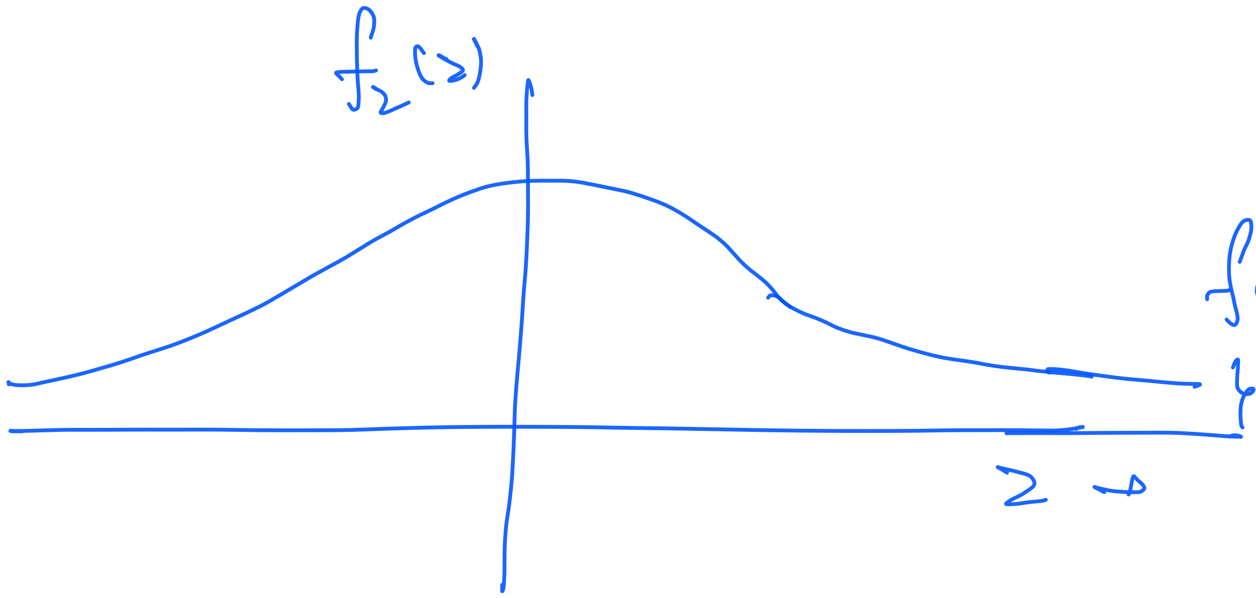
$$\Rightarrow (z^2+1)\frac{2y\,dy}{2} = dt$$

$$\Rightarrow y\,dy = \frac{dt}{1+z^2}.$$

$$\therefore f_z(z) = \frac{1}{\pi(1+z^2)} \int_0^{\infty} e^{-t}\,dt$$

$$= \frac{1}{\pi(1+z^2)} \left. \frac{e^{-t}}{-1} \right|_0^{\infty}$$

$$\left\{ f_z(z) = \frac{1}{\pi(1+z^2)} \right\} \longrightarrow \text{"Cachy distribution"}$$



fat tail as compare to gaussian, since cachy is decreasing quadratically.

Q.6) Let X_1, X_2 be two independent Random Variables both uniformly distributed between $[0,1]$. Find and sketch the cdf of $Y = \dfrac{\max(X_1, X_2)}{\min(X_1, X_2)}$.

Sol. 6)

We know, $\max(X_1, X_2) \geq \min(X_1, X_2)$

$$\therefore \boxed{Y \geq 1}$$

Now, to find (first):

$$F_Y(y) = \Pr(Y \leq y)$$

Let $B = \{Y \leq y\}$

and $A_1 = \{X_2 > X_1\}$

$A_2 = \{X_2 < X_1\}$

$A_3 = \{X_2 = X_1\}$

Then, $A_1 \cup A_2 \cup A_3 = \Omega$.

and, $P(A_1 \cup A_2 \cup A_3) = P(\Omega) = 1$

$\Rightarrow P(A_1) + P(A_2) + P(A_3) = 1$

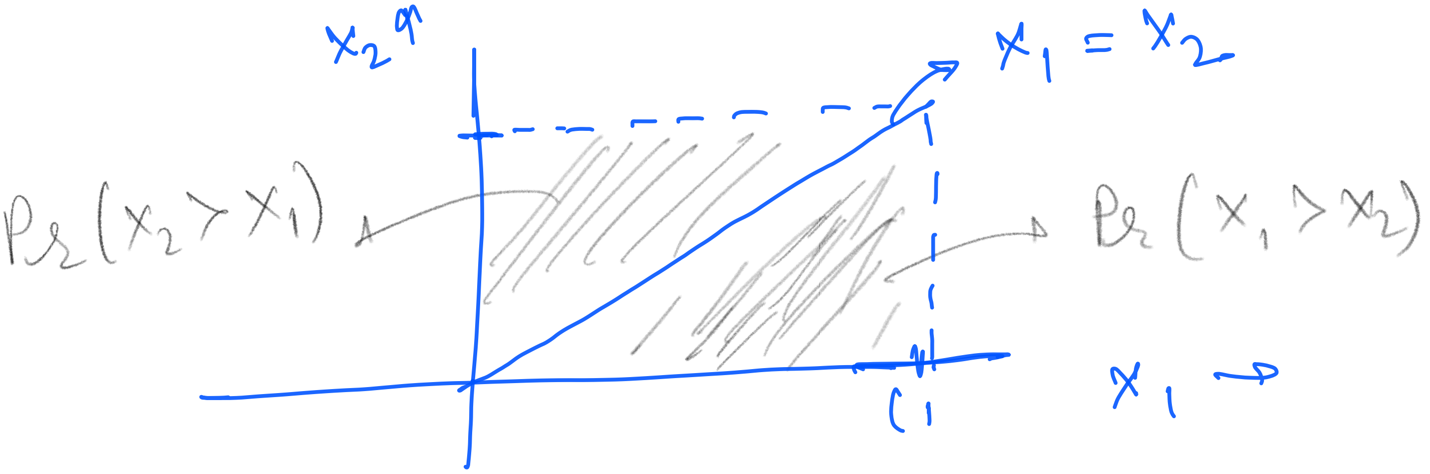
and, $P(A_3) = P(\{X_2 = X_1\}) = 0$

{since X_2, X_1 are continuous R.V.}

$\Rightarrow P(A_1) + P(A_2) = 1$

or

$$P(X_2 > X_1) + P(X_2 < X_1) = 1$$



$\therefore P(X_2 > X_1) = P(X_2 < X_1)$

$\Rightarrow 2P(X_2 > X_1) = 1$

$\Rightarrow P(X_2 > X_1) = 1/2$.

Now, let $A = X_2 > X_1$

then $\bar{A} = X_2 < X_1$

Then,

$$B \cap \Omega = B$$

$$\Rightarrow B \cap (A \cup \bar{A}) = (B \cap A) \cup (B \cup \bar{A})$$

(disjoint sets)

Thus,

$$F_Y(y) = Pr(Y \leq y)$$

$$= Pr(Y \leq y, X_2 > X_1)$$

$$+ Pr(Y \leq y, X_1 < X_2)$$

Now, if

$$P(Y \leq y, X_2 > X_1)$$

$$= P\left(\frac{X_2}{X_1} \leq y, X_2 > X_1\right)$$

$$= P(X_2 \leq X_1 y, X_2 > X_1)$$

$$= P(X_1 < X_2 \leq X_1 y)$$

and,

$$P(Y \le y,\ X_2 < X_1)$$

$$= P\left(\frac{X_1}{X_2} \le y,\ X_2 < X_1\right)$$

$$= P(X_1 \le X_2 y,\ X_2 < X_1)$$

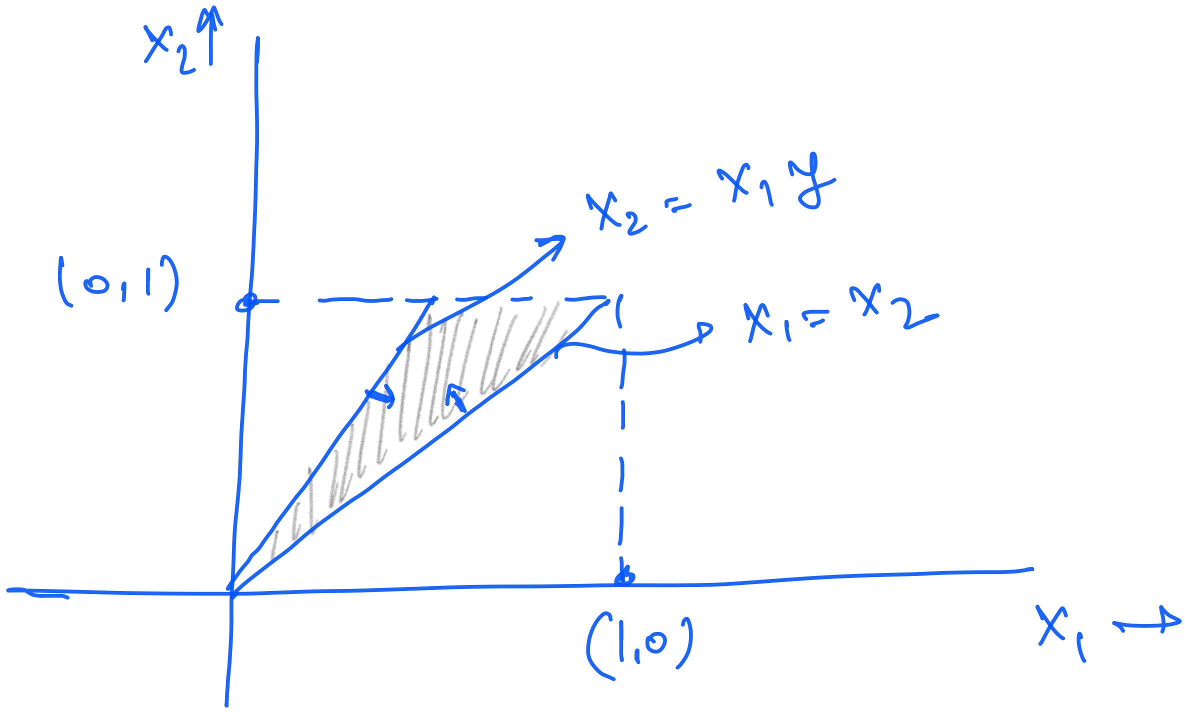
$$= P(X_2 < X_1 \le X_2 y)$$

$\because$ X_1 and X_2 are identical distribution

$$\therefore\ Pr(Y \le y,\ X_2 > X_1) = Pr(Y \le y,\ X_2 < X_1)$$

So, $$F_Y(y) = 2\,Pr(X_1 < X_2 \le X_1 y) \quad \text{--- (I)}$$

Now, sketching the region of integration for $\{X_1 < X_2 \le X_1 y\}$



Now, $\because y \geq 1$ So, $x_1 y \geq 1$

but x_2 has to ≤ 1 (since is uniform)

Now, let $C = \{x_1 < x_2 \leq x_1 y\}$.

and $D = \{x_1 y \leq 1\}$.

$\bar{D} = \{x_1 y > 1\}$.

Now,

$$Pr(C) = Pr(C \cap \Omega)$$
$$= Pr(C \cap (D \cup \bar{D}))$$

$$= \Pr(C \cap D) + \Pr(C \cap \bar{D})$$

$$= \Pr\left(\{X_1 < X_2 \le X_1 y\} \cap \{X_1 y \le 1\}\right)$$

$$+ \Pr\left(\{X_1 < X_2 \le X_1 y\} \cap \{X_1 y > 1\}\right)$$

$$= \Pr\left(\{X_1 < X_2 \le X_1 y\}, \{X_1 \le \tfrac{1}{y}\}\right)$$

$$+ \Pr\left(\{X_1 < X_2 \le 1\}, \{X_1 > 1/y\}\right)$$

$$= \int_0^{1/y} \int_{x_1}^{x_1 y} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1$$

$$+ \int_{1/y}^{1} \int_{x_1}^{1} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1$$

$$= \int_0^{1/y} \int_{x_1}^{x_1 y} f_{X_1}(x_1) \cdot f_{X_2}(x_2) \, dx_2 \, dx_1$$

$$+ \int_{1/y}^{1} \int_{x_1}^{1} f_{X_1}(x_1) \cdot f(x_2) \, dx_2 \, dx_1$$

$$\left(\because X_2 \perp\!\!\!\perp X_1 \right)$$

$$= \int_0^{1/y} x_1 (y-1) \, dx_1 + \int_{1/y}^{1} (1 - x_1) \, dx_1$$

$$= (y-1) \left. \frac{x_1^2}{2} \right|_0^{1/y} + \left. \frac{(1-x_1)^2}{-2} \right|_{1/y}^{1}$$

$$= \frac{(y-1)}{2} \frac{1}{y^2} + \frac{1}{2} \left(1 - \frac{1}{y} \right)^2$$

$$= \frac{y-1}{2y^2}\,(1 + y - 1)$$

$$= \frac{1}{2}\,\frac{(y-1)}{y}$$

$$= \frac{1}{2}\left(1 - \frac{1}{y}\right)$$

Thus,

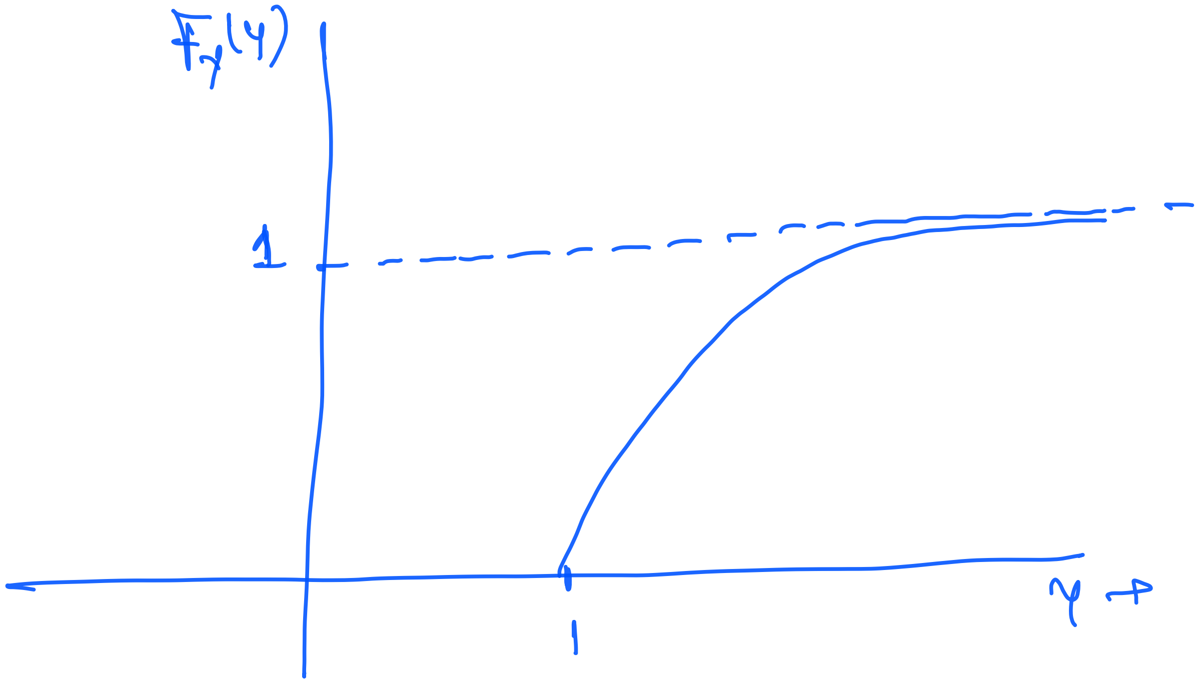
$$\Pr(\{X_1 < X_2 \le X_1 y\}) = \frac{1}{2}\left(1 - \frac{1}{y}\right)$$

So, from eq. (1),

$$F_Y(y) = \Pr(Y \le y)$$

$$= 2\Pr(X_1 < X_2 \le X_1 y)$$

$$\left\{ F_Y(y) = 1 - \frac{1}{y} \right\}, \quad \text{and} \quad y \geq 1$$



$F_Y(y)$
1
1
y →

Q 7) Let X, Y be two continuous R.V.s then, for R.V. XY then its pdf ?

Sol^n: Let $Z = XY$

So, $F_Z(z) = \Pr(Z \le z)$

$$= \Pr(XY \le z)$$

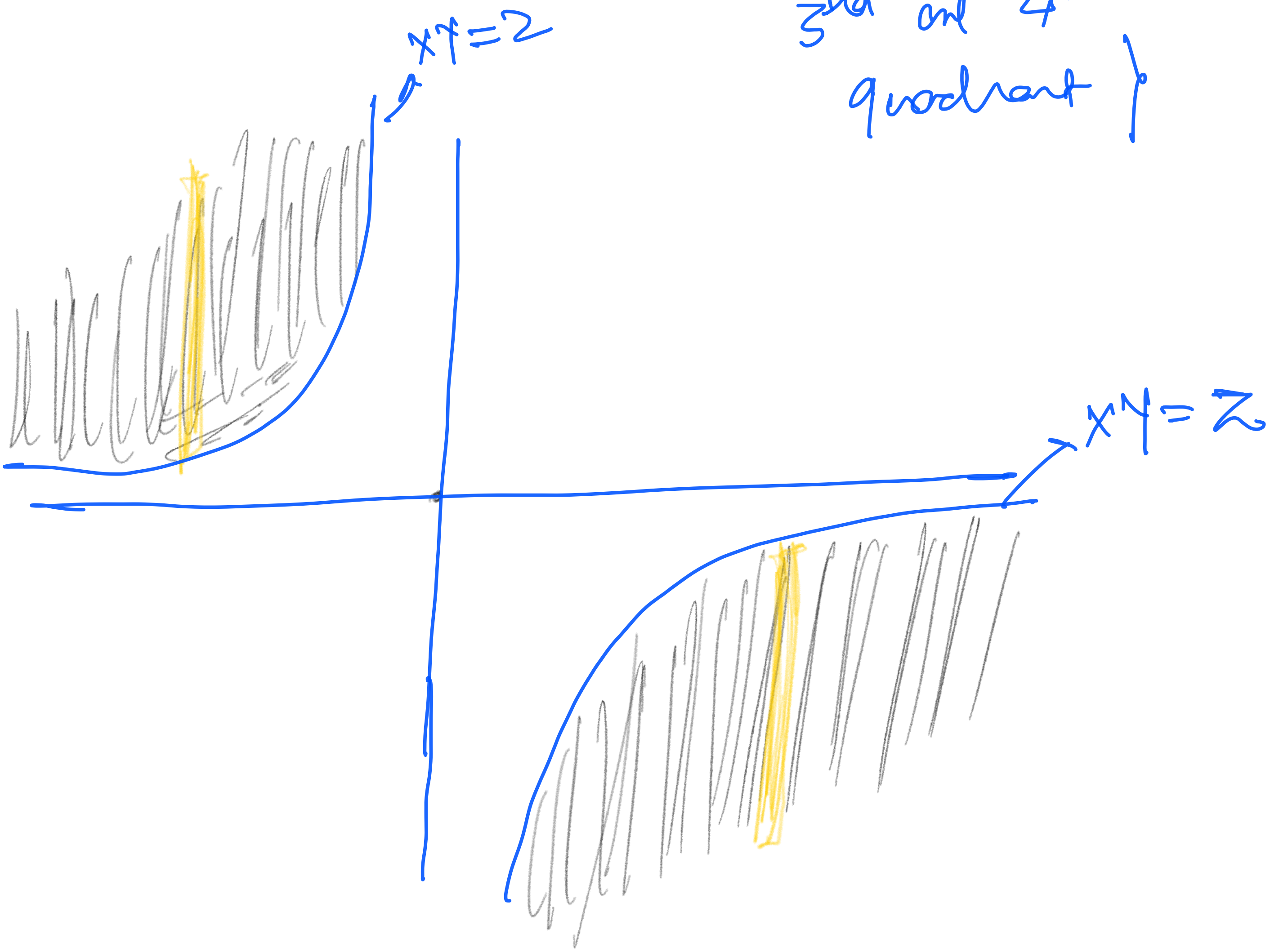
$$= \Pr\big(XY \le z \cap (X > 0 \cup X < 0)\big)$$

$$= \Pr\big((XY \le z) \cap (X > 0)\big) + \Pr\big((XY \le z) \cap (X < 0)\big)$$

$$= \Pr\left(\left(Y \le \frac{z}{X}\right), X > 0\right) + \Pr\left(\left(Y \ge \frac{z}{X}\right), (X < 0)\right)$$

Case (i) $z < 0$ → then $xy = z$

{ hyperbola in 2nd and 4th quadrant }



for $x > 0$,

take $(0,0)$ which doesn't follow $y \le \frac{z}{x}$

(for finding region)

since $0 \nleq -\frac{1}{0}$

$0 \nleq -\infty$

for $x > 0$,

(for finding region) take point $(-\infty, \infty)$

then, $Y \geq \frac{Z}{X}$ is satisfied.

$$F_Z(z) = \int_0^{\infty} \int_{-\infty}^{z/x} f_{XY}(x, y)\, dy\, dx + \int_{-\infty}^{0} \int_{z/x}^{\infty} f_{XY}(x, y)\, dy\, dx$$

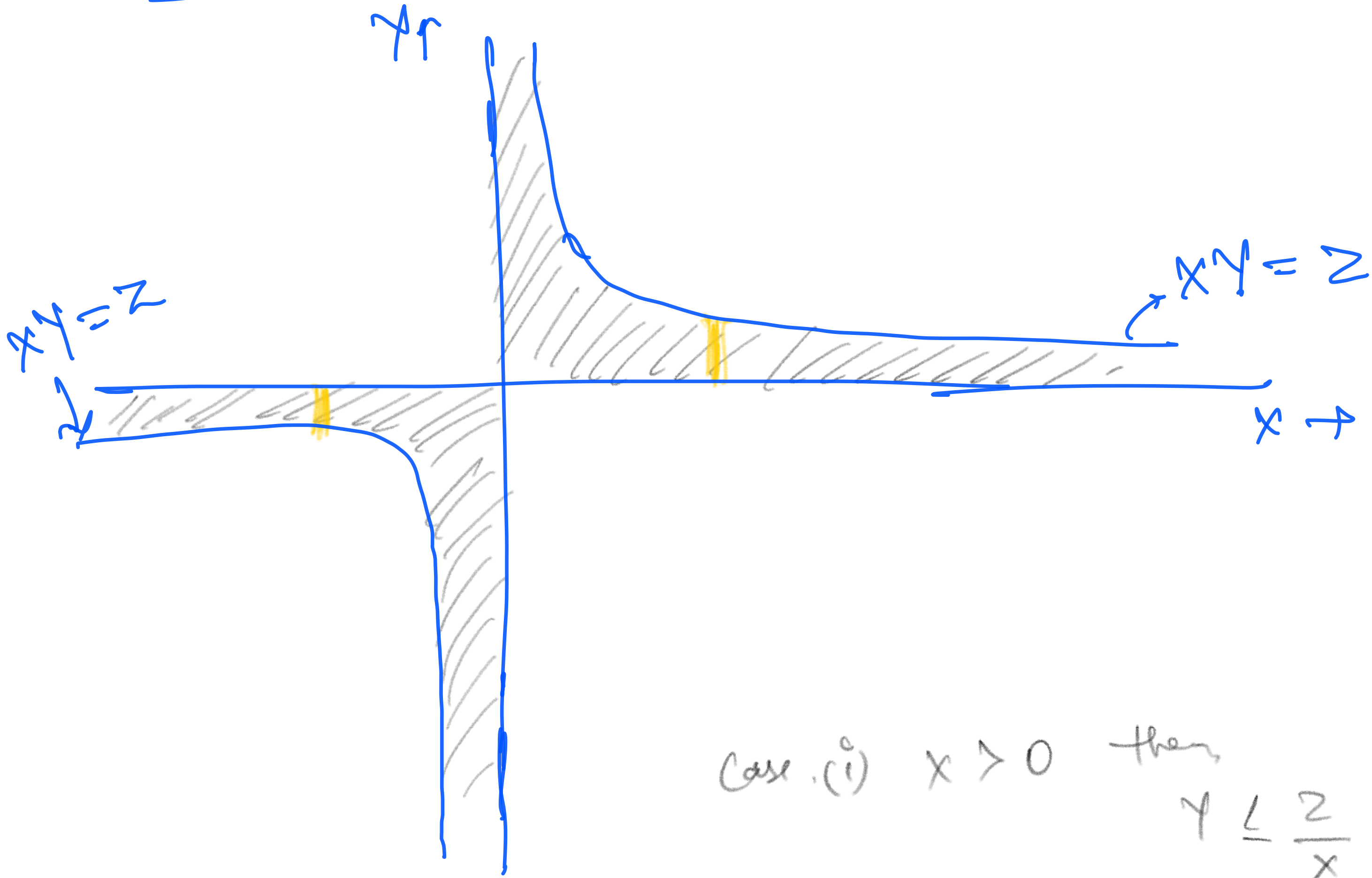
$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Then,

$$f_Z(z) = \int_0^{\infty} \frac{1}{x} f_{XY}\left(x, \frac{z}{x}\right) dx + \int_{-\infty}^{0} -\frac{1}{x} f_{XY}\left(x, \frac{z}{x}\right) dx$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{XY}\left(x, \frac{z}{x}\right) dx$$

Case (ii) $z \geq 0$.



Case (i) $X > 0$ then, $Y \leq \frac{z}{X}$

here $(0,0)$ satisfies it.

Case (ii) $X < 0$ then, $Y \geq \frac{z}{X}$

here also $(0,0)$ satisfies it.

Then,

$$F_Z(z) = \Pr\left(Y \leq \frac{z}{X}, X > 0\right) + \Pr\left(Y \geq \frac{z}{X}, X < 0\right)$$

$$= \int_0^\infty \int_0^{z/x} f_{XY}(x,y)\,dy\,dx + \int_{-\infty}^0 \int_{z/x}^0 f_{XY}(x,y)\,dy\,dx$$

Then, same as above we will get

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_{-\infty}^\infty \frac{1}{|x|} f_{XY}(x, z/x)\,dx.$$

$$f_Z(z) = \int_{-\infty}^\infty \frac{1}{|x|} f_{XY}\left(x, \frac{z}{x}\right) dx \quad \forall z \in \mathbb{R}.$$

Q. 8) $Z \sim Ber(\frac{1}{2})$ and $Y \sim N(0,1)$

$Y \perp\!\!\!\perp Z$.

What is CDF of YZ = ?

$$F_X(x) = Pr(X \le x)$$

$$= Pr(YZ \le x)$$

$$= Pr(YZ \le x \cap (Z=0 \cup Z=1))$$

$$= Pr(YZ \le x, Z=0) + Pr(YZ \le x, Z=1)$$

$$= Pr(YZ \le x \mid Z=0) \cdot P(Z=0) + Pr(YZ \le x \mid Z=1) \cdot P(Z=1)$$

$$= \Big(\underbrace{Pr(0 \le x)}_{\text{cdf of '0' constant R.V. function}} + \underbrace{Pr(Y \le x)}_{= F_Y(x)}\Big)\frac{1}{2}.$$

Case (i) if $x \geq 0$

then $\Pr(0 \leq x) = 1$

Case (ii) if $x < 0$

then, $\Pr(0 \leq x) = 0$.

$\therefore$

$$F_X(x) = \left(\mathbb{1}\{x : x \geq 0\} + F_X(x) \right) \frac{1}{2}$$

(My personal suggestions)

A few resources where you can solve more questions on probability (other than what book sir suggested)

- Stats 110 (Harvard course)
  - ↳ it has book, handouts, problems.
- probabilitycourse.com
- Questions from IISc previous offering / NPTEL probability course questions.

> This list is not exhaustive, just my recommendation.