Q2. $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ (To show bijective)

Hopcroft and Ulmann pairing function
$\underbrace{\text{(derived from Cantor's pairing function)}}$

$$f(m,n) = \frac{1}{2}(m+n-2)(m+n-1) + n$$

(A) $\boxed{\text{Injective}}$ if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

So let,

$$f(m,n) = f(p,q) \quad \text{——} \quad (1)$$

assume that, (for contradiction),

$$(m,n) \neq (p,q) \text{ and}$$

$$m+n < p+q \quad (\text{WLOG})$$

Now, let

$$m+n-2 = a$$

$$d = (p+q-2) - (m+n-2)$$

Now, $\because$ $p+q > m+n$

$$\Rightarrow p+q-2 > m+n-2, \quad (\because m,n,p,q \in \mathbb{N})$$

$$\Rightarrow (p+q-2) - (m+n-2) \geq 1$$

So, $\boxed{d \geq 1}$

Now, from (I),

$$\frac{1}{2} a(a+1) + n = \frac{1}{2}(d+a)(d+a+1) + q \quad \text{— (II)}$$

$$n - q = \frac{1}{2}\Big((d+a)(d+a+1) - a(a+1)\Big)$$

$$= \frac{1}{2}\Big(d(d+a+1) + ad + a(a-1) - a(a+1)\Big)$$

$$n - q = ad + \frac{d(d+1)}{2}$$

$$n - q \geq a + 1 \quad (\because d \geq 1)$$

$$\Rightarrow n \geq q + a + 1 = q + m + n - 1 > n$$

$$(\because m, q \in \mathbb{N})$$

So, $\underline{n > n}$ which is wrong,

thus, the contradiction of our assumption,

So, $m + n = p + q$,

$\downarrow$

from, (ii), we get

$\boxed{n = q} \Rightarrow \boxed{m = p}$ (so, injective)

(B) <u>Surjective</u>

$\forall y \in$ codomain of f, $\exists x \in \text{dom}(f)$
s.t $f(x) = y$.

and if we show that $f^{-1}(y) = \{x\}$ then function 'f' is invertible i.e. (one-one and onto) both.

So, let,

$$z = f(m,n) = \frac{1}{2}(m+n-2)(m+n-1) + n$$

$\forall m, n, z \in \mathbb{N}$.

and let,

$$\left[\begin{array}{l} w = m+n-2 \\ t = \dfrac{w(w+1)}{2} \\ z = t+n \end{array}\right] \begin{array}{l} \rightarrow w \geq 0 \text{ (a whole number)} \\ \rightarrow t \geq 0 \end{array}$$

So, if we can write each unique w as a function of unique z, then, we can associate each z to unique m and n.

Now,

$$2t^2 = w(w+1)$$

$$\Rightarrow w^2 + w - 2t^2 = 0$$

$$w = \frac{-1 + \sqrt{1+8t}}{2}$$

and

$$z > t \quad (\because n, z \in \mathbb{N})$$

$$\Rightarrow z-1 \geq t$$

$$w = \frac{-1+\sqrt{1+8t}}{2}$$

So,

$$\boxed{w \leq \frac{-1+\sqrt{1+8(z-1)}}{2}}$$

Now,

$$\begin{aligned} t \leq z-1 &= t+n-1 \\ &< t+m+n-1 \quad (\because m \in \mathbb{N}) \\ &< \frac{1}{2}w(w+1) + m+n-1 \\ &< \frac{1}{2}w(w+1) + w+1 \end{aligned}$$

$$2(z-1) < w^2 + w + 2w + 2$$

$$2(z-1) < (w+1)^2 + (w+1)$$

$$\Rightarrow \quad (w+1)^2 + (w+1) - 2(z-1) > 0$$

$$\Rightarrow \quad \boxed{(w+1) > \frac{-1 + \sqrt{1+8(z-1)}}{2}}$$

$$\therefore \quad w \leq \frac{-1 + \sqrt{1+8(z-1)}}{2} < w+1$$

$\because$ w is a whole number.

$$w = \left\lfloor \frac{-1 + \sqrt{1+8(z-1)}}{2} \right\rfloor$$

($\lfloor\ \rfloor$ floor function) {greatest integer function}

Now, for $\forall z \in \mathbb{N}$,

we get w.

then $t = \frac{1}{2} w(w+1)$ (unique t)

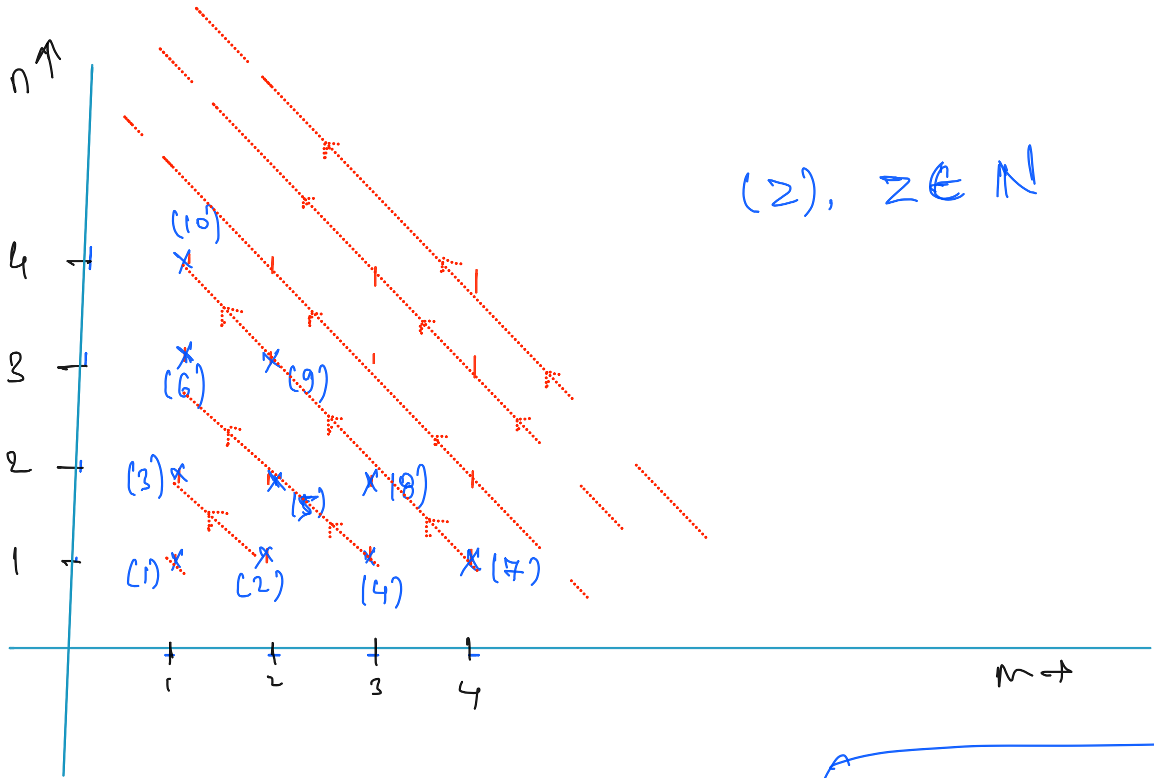
$\Rightarrow \quad n = z - t$ (unique n)

$\Rightarrow m = w - (n - z)$ (unique m)

Thus, function $f : \mathbb{N}^2 \to \mathbb{N}$, defined by

$$f(m,n) = \frac{1}{2}(m+n-2)(m+n-1) + n \quad \forall\, m,n \in \mathbb{N}$$

is invertible, so, its bijective!



$(z),\ z \in \mathbb{N}$

detour

Can also used in Countable union of Countable set is Countable

Now,

$$|\mathbb{N}^3| = |\mathbb{N}^2 \times \mathbb{N}|$$

we can replace $\mathbb{N}^2$ with $\mathbb{N}$ ($\because |\mathbb{N}^2| = |\mathbb{N}|$)

So, $|\mathbb{N}^3| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Now, using principle of Mathematical Induction [for any finite $d \in \mathbb{N}$]

Assuming $\mathbb{N}^{d-1}$ is bijective with $\mathbb{N}$.

Then,

$$|\mathbb{N}^d| = |\mathbb{N}^{d-1} \times \mathbb{N}|$$
$$= |\mathbb{N} \times \mathbb{N}|$$
$$= |\mathbb{N}^2| = |\mathbb{N}|$$

So, $\mathbb{N}^d$ is also bijective.

here d can't be $\mathbb{N}$., it is similar to the fact that union of finite sets is different from union of countably infinite sets, we can't just take $d \to \infty$.

Q.4

$f : A \to 2^A$

To show:
$$|A| < |2^A|$$

$A \in \{$ finite set, countable infinite set, uncountable set $\}$

Assume (for contradiction) that, $\exists f$, s.t. $f: A \to 2^A$ is bijective.

i.e. $|A| = |2^A|$

Let $A = \{a, b, c, \ldots\}$

$f(A) = \{f(a), f(b), f(c), \ldots\}$

Now, consider the following mapping:

D	a	b	c	e	g	$\cdots$
$f(a)$	a		c		g	$\cdots$
$f(b)$	a	b		e	g	$\cdots$
$f(c)$		b	c	e		$\cdots$
$f(e)$	a	b	c	e		$\cdots$
$f(g)$	a		c		g	$\cdots$
$\vdots$						

$f(a)$: any subset of A

$$\left(\begin{array}{l} a \to f(a) \\ b \to f(b) \\ \vdots \end{array}\right)$$

Consider a set

$$D = \{ b, c, e, \ldots \}$$

which is

$$D = \{ b \in A \mid b \notin f(b) \}$$

Now, $D \subseteq A$, so, if f is bijective, then D has to a unique pre-image in Domain.

i.e.

$$\exists\, x \in A, \text{ s.t. } f(x) = D.$$

Now,

Case (i) $x \in D$, then, $x \notin f(x) = D$,
which is a contradiction.

Case (ii) $x \notin D$, then $x \in f(x) = D$,
which is a contradiction,

thus,

$f : A \to 2^A$ is not surjective and

hence,

$$|2^A| > |A|.$$