Linear Strategic Classification with Endogenous Improvements

Siddharth Shrivastava*
Department of Artificial Intelligence
IIT Hyderabad
siddharthmanik16@gmail.com

Mahvith Akshintala*
Department of Artificial Intelligence
IIT Hyderabad
akshintalamahvith@gmail.com

B Vamsha Vardhan Reddy
IIIT Hyderabad
vamsha1401@gmail.com

Naresh Manwani
IIIT Hyderabad
naresh.manwani@iiit.ac.in

Sujit Gujar
IIIT Hyderabad
sujit.gujar@iiit.ac.in

Ganesh Ghalme
Departement of Artifical Intelligence
IIT Hyderabad
ganeshghalme@ai.iith.ac.in

Abstract

Strategic classification studies settings in which agents respond to a deployed classifier by modifying observable features at a cost. Classical models typically treat such responses as cosmetic: features may change, but true labels remain fixed. We study an improvement-aware variant in which strategic responses can induce genuine changes in outcome-relevant features. Agents choose post-deployment feature vectors strategically, and labels are then generated according to a stable conditional outcome law that preserves the relationship between features and outcomes.

We formalize this problem for linear classifiers under a single-index qualification model and linear-decomposable costs. We show that the strategic-optimal classifier is obtained by a parallel shift of the Bayes-optimal decision boundary, and that it provides a better surrogate for the improvement-aware objective than the Bayes classifier. Since improvement-aware learning requires post-deployment labels, which are typically unavailable before deployment, we provide PAC-style guarantees under an oracle model, propose a practical plug-in algorithm, establish its generalization bound, and evaluate it on synthetic and real-world datasets.

1 Introduction

Strategic classification [20] studies prediction settings in which agents respond to a deployed classifier by modifying observable features at a cost. Such behavior arises in credit scoring, admissions [20, 15], and hiring [24]: applicants may adjust credit utilization, obtain certifications, or enroll in test-preparation programs to improve their chances of receiving a favorable decision. Classical strategic classification treats these responses as manipulation: the feature vector changes, but the underlying label or qualification remains fixed.

This immutability assumption is often too restrictive. In many applications, the same actions that change observable features can also improve the underlying outcome of interest. Coursework may

*Equal contribution.

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improve academic readiness, professional training may increase job productivity, and financial discipline may improve repayment probability. Thus, a classifier can both induce strategic feature changes and incentivize genuine improvement. We call this setting *improvement-aware* strategic classification.

We study improvement-aware strategic classification with *endogenous improvement* for binary label prediction tasks in this paper. In particular, we assume that the original population is governed by a stable conditional outcome law $\eta(x) = \Pr(Y = 1 \mid X = x)$. The deployed classifier affects agents by changing their incentives over feasible feature modifications. Thus, the classifier changes the marginal distribution of features, but not the structural relation between features and outcomes. In this sense, improvement is endogenous: it is not externally imposed by the analyst, but arises because agents move to regions of the feature space that, under the original data-generating process, are associated with better outcomes.

The feature manipulation incurs a cost. The cost function captures the effort or resources required to move from original features to modified features. An agent's utility is then the difference between the benefit of a favorable classification outcome and the cost incurred to achieve it. For instance, a loan applicant weighs the benefit of credit approval against the cost of temporarily manipulating credit utilization, while a job candidate balances the value of an offer against the expense and effort of obtaining certifications. By explicitly modeling these cost-benefit calculations, the framework enables analysis of how agents will strategically respond to any given classifier.

Our framework makes two assumptions: 1) the class-probability function η follows a single-index qualification model, $\eta(x) = g(w^{\star\top} x)$, for a non-decreasing link $g : \mathbb{R} \to [0, 1]$ and non-negative weights $w^\star \in \mathbb{R}^d_{\geq 0}$, and 2) the manipulation cost is linear-decomposable, i.e., the total cost is a weighted sum of feature-wise improvement costs. Assumption 1 guarantees that non-strategic Bayes optimal classifier is linear, and subsumes the monotonicity of η in features. Monotonicity of the outcome in improvement-relevant features is a standard assumption in the improvement-aware strategic classification literature [24, 28]. It also encompasses many class-probability models commonly used in the literature, including linear and sigmoidal models [6, 20, 38]. We will discuss these assumptions in detail in Section 2.

Our Results and Paper Organization Section 2 introduces the model and assumptions. Section 3 gives the structural results: Theorem 3.2 shows that the strategic-optimal linear classifier is a parallel shift of the Bayes classifier, and Theorem 3.4 shows that this shifted classifier is a better proxy for the improvement-aware objective than the Bayes classifier. Section 4 studies learnability: Theorem 4.2 gives a uniform-convergence guarantee under sampled post-response labels, and Theorem 4.4 establishes a plug-in generalization bound for STRAT-IMP-AWARE. Section 5 reports experiments on synthetic and real-world datasets.

1.1 Related Work

Hardt et al. [20] formalized Strategic Classification as a Stackelberg game between learner and agents who manipulate observable features at a cost. Subsequent works explore several aspects of strategic classification including the social cost of manipulation [29], information asymmetry between learner and agents [18, 32], sequential version of strategic classification [12] and so on. Kolkman et al. [25] develops a differentiable surrogate for strategic response that enables end-to-end gradient based training. Classical strategic classification treats manipulation as costly feature movement that leaves the underlying label unchanged– an assumption our framework relaxes.

Strategic classification has been extended to settings where agents can genuinely improve rather than gaming the classification rule by feature-level manipulation. This setting is closely related to performative prediction problem introduced by [30]. One thread studies classifier design that incentivizes productive effort over gaming [24, 23, 2, 27]. Chen et al. [11] studies incentivizing improvement under deterministic qualification improvement with tuned surrogate objective, and Xie et al. [37] give guarantees for 1-dimensional threshold classifier under deterministic relations with active experimentation at deployment; both require deterministic assumptions that our probabilistic, classifier-endogenous framework relaxes. A second thread analyzes fairness under improvement-aware response [16, 4]. A third line of work in this setting grounds the improvement/gaming distinction in causal structure on features [28, 21, 17]. Finally, several studies extend the improvement-aware analysis to dynamics, persistent improvement, and welfare [19, 8, 36, 22]. Our framework treats improvement as emergent from a probabilistic, classifier-endogenous data-generating process.

A line of work develops learning-theoretic foundations for strategic classification under best-responding agents, including PAC-learnability and strategic VC-dimension results [34], incentive-aware sample complexity bounds [39], regret bounds in the online setting [3], and learnability gaps between strategic and standard PAC learning [13]. A recent thread extends this analysis to settings with genuine improvement: Attias et al. [5] prove PAC bounds under exogenously specified improvement sets, Sharma and Sun [33] analyze linear classifiers with bounded improvement. These two close works to ours, Attias et al. [5] and Sharma and Sun [33], both operate under deterministic labels and worst-case tie-breaking, with improvement sets that are either exogenous geometric objects or bounded regions independent of the classifier. Our setting relaxes both assumptions — the label is probabilistic and the manipulation region is classifier-dependent — yielding PAC learning guarantee (Theorem 4.2) and generalization guarantee (Theorem 4.4) with no direct analogue in either work.

Prior work relies on exogenous structure: causal graphs, binary improvable/non-improvable feature labels, or bounded improvement sets to separate manipulation from improvement, requiring strong domain knowledge often unavailable in practice. Our framework resolves this dichotomy: every manipulation induces a probabilistic label shift through a class-probability function, with the magnitude of improvement determined endogenously by the deployed classifier through agent utility maximization under decomposable costs.

2 Settings and Preliminaries

Let $\mathcal{X} \subseteq \mathbb{R}^d$ denote the feature space, and $\mathcal{D}$ be a distribution over $\mathcal{X}$. Also let $\mathcal{X}_0 := \mathrm{supp}(\mathcal{D}) \subseteq \mathcal{X}$ denote the support of the original feature distribution. Agents may report feature vectors in a feasible report set $\mathcal{A} \subseteq \mathcal{X}$, with $\mathcal{X}_0 \subseteq \mathcal{A}$. For simplicity we take $\mathcal{A}$ to be common across agents; the model can also allow agent-dependent feasible sets $\mathcal{A}(x) \subseteq \mathcal{X}$. We define the class-probability function as $\eta(x) = \mathbb{P}(y = 1|x)$, where the label for x is drawn as $y_x \sim \mathrm{Bernoulli}(\eta(x))$. The training dataset D consists of unaltered points $(x_i, y_{x_i})_{i=1}^n$, where the x_i's are independently sampled from $\mathcal{D}$ and y_{x_i} are independent realizations of $\eta(x_i)$. Wherever x_i is clear from the context, we will write y_i to denote y_{x_i} for notational simplicity. Further, we define $\mathtt{err}(\cdot)$ as the standard expected $0-1$ loss, $\mathtt{err}(f) = \mathbb{E}_{x\sim\mathcal{D}}[\mathbb{1}(f(x) \neq y_x)]$.

Definition 2.1 (Agent Best Response)**.** *For a classifier f, an agent with original feature vector $x \in \mathcal{X}_0$ reports*

$$x^f \in \arg\max_{z\in\mathcal{A}} \{\beta f(z) - c(x, z)\},$$

where $\beta > 0$ is the benefit from positive classification and $c(x, x')$ is the cost of changing features from x to x'.

Strategic reporting induces a mismatch between the training distribution and the distribution observed at deployment. A strategically robust classifier accounts for this best-response-induced distribution shift and adjusts its decision rule accordingly. The goal in standard strategic classification setting is to minimize the strategic error given below.

Definition 2.2 (Strategic Error)**.** *The strategic error of a classifier $f \in \mathcal{F}$ is defined as*

$$\mathtt{err}_s(f) = \mathbb{E}_{x\sim\mathcal{D}}\left[\mathbb{1}[f(x^f) \neq y_x]\right]. \tag{1}$$

This definition implicitly assumes labels are immutable. We modify the above objective to reflect the resulting change in qualification/labels in improvement-settings as follows.

Definition 2.3 (Improvement-Aware Strategic Error)**.** *The improvement-aware strategic error of a classifier $f \in \mathcal{F}$ is defined as*

$$\mathtt{err}_{\mathrm{Imp}}(f) = \mathbb{E}_{x\sim\mathcal{D}}\left[\mathbb{1}[f(x^f) \neq y_{x^f}]\right]. \tag{2}$$

We consider endogenous improvement in qualification. This modeling choice captures strategic behavior as inducing true qualification changes. In particular, the probabilistic gain in qualification is a function of distance from classifier boundary, in contrast to models that either partition agents exogenously into gaming versus improving types [21],[11] or treat improvement as deterministic [5]. To characterize when and how feature improvements translate to qualification improvements, we assume the following condition on the model.

Assumption 1 (Single-index qualification model)**.** *There exists a continuous non-decreasing function* $g : \mathbb{R} \to [0, 1]$ *and a vector* $w^\star \in \mathbb{R}^d_{\geq 0}$ *such that* $\eta(x) = g((w^\star)^\top x)$ *for all* $x \in \mathcal{X}$*. Moreover, the Bayes threshold* $b^\star := \inf\{t \in \mathbb{R} : g(t) \geq 1/2\}$ *is finite.*

The single-index qualification model ensures that the true data-generating process respects a natural alignment between feature improvements and qualification improvements, enabling us to design classifiers that leverage this structure. This class of models covers several standard probabilistic classification models, including logistic, probit, and other monotone-link generalized linear models; the sigmoid link is a common example [25]. Furthermore, Assumption 1 is particularly justified in domains where features represent positive attributes, skill investments, or substantive credentials rather than mere signals. We validate this assumption on real-world datasets in Appendix G. Next we observe that Assumption 1 guarantees that the optimal classifier is linear.

Observation 2.4. *[Existence of a linear Non-strategic Bayes optimal classifier] Under* 0–1 *loss in the Single-index qualification model (Assumption 1), there exists a Non-strategic Bayes optimal classifier of the form* $f^\star(x) = \mathbf{1}\{{w^\star}^\top x \geq b^\star\}$ *for some threshold* $b^\star \in \mathbb{R}$*. In particular, if* $b^\star := \inf\{t \in \mathbb{R} : g(t) \geq 1/2\}$*, then the classifier above is Bayes-optimal.*

This observation (proof in Appendix B) allows us to restrict the space of classifiers to the set of linear classifiers with non-negative weights and hence, throughout the paper, we will assume that

$$\mathcal{F} := \{f_{w,b} : \mathcal{X} \to \{0, 1\} : f_{w,b}(z) = \mathbb{1}\{w^\top z \geq b\},\ w_i \geq 0\ \forall i \in [d],\ b \in \mathbb{R}\}.$$

Definition 2.5 (Linear-decomposable cost)**.** *A cost function* $c : \mathcal{X}_0 \times \mathcal{A} \to \mathbb{R}_+$ *is decomposable if there exist* $\alpha_i > 0$ *for* $i \in [d]$ *such that*

$$c(x, x') = \sum_{i=1}^{d} \alpha_i (x'_i - x_i)_+.$$

Here $(x'_i - x_i)_+ = \max(0, x'_i - x_i)$.

The cost model is a weighted one-sided ℓ_1 action cost, where the parameters α_i denotes the per-unit cost or difficulty of improving a given feature i. It is closely related to the linear/separable manipulation costs used in strategic classification [20, 1] and to coordinate-wise action costs used in algorithmic recourse [35]. This structure captures settings where each feature can be improved independently at some per-unit cost for example, hours spent on test preparation, tuition for certification courses, or effort invested in building work experience.

Assumption 2. *The cost function is linear-decomposable.*

First, we prove certain structural results for the improvement-aware strategic classification.

3 Structural Results

Our first result shows that if a non-strategic classifier has non-negative weights, then the agents are incentivized to improve their feature values, i.e., the agents' best-response feature vector is coordinate-wise greater than the original feature vector.

Lemma 3.1. *Suppose* $\mathcal{A}$ *is coordinate-wise order-convex: for every* $x \in \mathcal{X}_0$*,* $z \in \mathcal{A}$*, and every* $\tilde{z}$ *lying coordinate-wise between* x *and* z*, we have* $\tilde{z} \in \mathcal{A}$*. Let* $f_{w,b} \in \mathcal{F}$ *be a linear classifier with* $w \in \mathbb{R}^d_{\geq 0}$ *and* $b \in \mathbb{R}$*. Then, for every* $x \in \mathcal{X}_0$*, there exists a strategic best response* $x^f \in \mathcal{A}$ *satisfying* $x^f_i \geq x_i$ *for all* $i \in [d]$*.*

The resultant feature improvement induces the improvement in the qualification, which is a central theme of the paper. We are now ready to prove the first important result of the paper.

Theorem 3.2. *Let* $\mathcal{F}$ *be the class of linear classifiers* $f_{w,b}(x) = \mathbf{1}\{w^\top x \geq b\}$*, where* $w \in \mathbb{R}^d_{\geq 0}$*,* $w \neq 0$*, and* $b \in \mathbb{R}$*. Let the cost function be linear decomposable, i.e.,* $c(x, x') = \sum_{i=1}^d \alpha_i (x'_i - x_i)_+$*, where* $\alpha_i > 0$*. For a given classifier* $f_{w,b} \in \mathcal{F}$ *parameterized by* w *and* b*, define* $r(w) := \max_{i \in [d]} \frac{w_i}{\alpha_i}$ *and* $b_s := b + \beta r(w)$*, and let* $f_s(x) = \mathbf{1}\{w^\top x \geq b_s\}$*. Assume that for some* $i^\star \in \arg\max_{i \in [d]} w_i/\alpha_i$*, every required upward move* $x + t e_{i^\star}$ *remains feasible in* $\mathcal{A}$ *for all* $x \in \mathcal{X}_0$ *and relevant* $t \geq 0$*; in* *particular, this holds when $\mathcal{A} = \mathcal{X} = \mathbb{R}^d$. Assume also that agents select a tie-broken best response, with ties resolved in favor of positive classification whenever a positive best response exists. Then, for every $x \in \mathcal{X}_0$, the selected best response x^{f_s} satisfies*

$$f_s(x^{f_s}) = f_{w,b}(x).$$

Consequently, $\mathtt{err}_s(f_s) = \mathtt{err}(f_{w,b})$.

Proof. Fix $f_{w,b} \in \mathcal{F}$ and let $b_s = b + \beta r(w)$. Let x^{f_s} denote the selected tie-broken best response to the shifted classifier f_s. For any selected report z in the range of the best-response map, define $\mathcal{Z}_z := \{x \in \mathcal{X}_0 : z \text{ is the selected best response to } f_s \text{ for agent } x\}$. These sets partition $\mathcal{X}_0$ according to the selected best-response rule. It is therefore enough to show that, for every selected report z and every $x \in \mathcal{Z}_z$, $f_s(z) = f_{w,b}(x)$.

We first record a useful bound. For any $x \in \mathcal{X}_0, y \in \mathcal{A}$,

$$w^\top(y-x) \le \sum_{i=1}^{d} w_i(y_i - x_i)_+ \le r(w)\sum_{i=1}^{d} \alpha_i(y_i - x_i)_+ = r(w)c(x,y). \tag{3}$$

We now consider three cases.

Case 1: $w^\top z > b_s$**.** Then $f_s(z) = 1$. Suppose, for contradiction, that $f_{w,b}(x) = 0$ for some $x \in \mathcal{Z}_z$. Then $w^\top x < b$, and hence $f_s(x) = 0$. Since staying at x gives utility 0, while reporting z gives utility $\beta - c(x,z)$, optimality of z implies $c(x,z) \le \beta$. Using the bound above, $w^\top z \le w^\top x + r(w)c(x,z) \le w^\top x + \beta r(w) < b + \beta r(w) = b_s$, contradicting $w^\top z > b_s$. Hence $f_{w,b}(x) = 1 = f_s(z)$.

Case 2: $w^\top z = b_s$**.** Again $f_s(z) = 1$. Suppose, for contradiction, that $f_{w,b}(x) = 0$ for some $x \in \mathcal{Z}_z$. Then $w^\top x < b$ and $f_s(x) = 0$. Since z is a selected best response, we must have $c(x,z) \le \beta$. Therefore, $b_s = w^\top z \le w^\top x + r(w)c(x,z) \le w^\top x + \beta r(w) < b + \beta r(w) = b_s$, a contradiction. Thus $f_{w,b}(x) = 1 = f_s(z)$.

Case 3: $w^\top z < b_s$**.** Then $f_s(z) = 0$. Suppose, for contradiction, that $f_{w,b}(x) = 1$ for some $x \in \mathcal{Z}_z$, i.e., $w^\top x \ge b$. If $w^\top x \ge b_s$, then staying at x gives positive classification with zero cost and utility β, whereas reporting z gives utility $-c(x,z) \le 0$, contradicting optimality of z. Hence we may assume $b \le w^\top x < b_s$. Let $\delta := b_s - w^\top x \in (0, \beta r(w)]$. Choose $i^\star \in \arg\max_i w_i/\alpha_i$. Since $w \ne 0$ and $\alpha_i > 0$, we have $w_{i^\star} > 0$. By the feasibility assumption, the point $y := x + \frac{\delta}{w_{i^\star}} e_{i^\star}$ belongs to A. Moreover, $w^\top y = b_s$, so $f_s(y) = 1$, and $c(x,y) = \alpha_{i^\star}\frac{\delta}{w_{i^\star}} = \frac{\delta}{r(w)} \le \beta$. Thus reporting y gives utility $\beta - c(x,y) \ge 0$. On the other hand, since $f_s(z) = 0$, reporting z gives utility $-c(x,z) \le 0$. If $\beta - c(x,y) > 0$, this contradicts optimality of z. If $\beta - c(x,y) = 0$, then y is a positive best response with utility equal to that of any zero-utility negative response, so the tie-breaking rule selects a positive response rather than z. This again contradicts z being the selected report. Therefore $f_{w,b}(x) = 0 = f_s(z)$.

Combining the three cases, for every selected report z and every $x \in \mathcal{Z}_z$, $f_s(z) = f_{w,b}(x)$. Equivalently, for every $x \in \mathcal{X}_0$, $f_s(x^{f_s}) = f_{w,b}(x)$. Taking expectation over the data distribution yields $\mathtt{err}_s(f_s) = \mathtt{err}(f_{w,b})$. □

Corollary 3.3. *Suppose the assumptions of Theorem 3.2 hold for every linear classifier in $\mathcal{F}$, including the feasibility and tie-breaking assumptions. Let $f^\star$ denote the Non-strategic Bayes optimal linear classifier with parameters $(w^\star, b^\star)$. Then the shifted classifier $f_s^\star(x) = \mathbf{1}\{(w^\star)^\top x \ge b^\star + \beta \max_i w_i^\star/\alpha_i\}$ is strategic-optimal among linear classifiers.*

An important consequence of Corollary 3.3 is that the strategically optimal classifier has a simple structure and can be implemented via the Non-strategic Bayes optimal classifier, without explicitly learning strategic responses from data. In contrast, designing an optimal *improvement-aware* classifier is substantially more challenging. As local manipulation region depends on the classifier, the observed, pre-deployment data provide no information about post-implementation labels. This makes it difficult to anticipate improvement effects and incorporate them into classifier design. Our next result compares relative performance guarantees of Non-strategic Bayes optimal and strategic optimal classifiers with respect to improvement-aware objective.

Theorem 3.4. *Under Assumptions 1 and 2, the feasibility and tie-breaking conditions of Theorem 3.2, and the canonical rule that among positive best responses agents choose a minimum-cost positive report, the strategic-optimal shifted classifier $f_s^\star$ satisfies*

$$err_{\text{Imp}}(f_s^\star) \le err_{\text{Imp}}(f^\star).$$

Proof. Let $Z = w^{\star\top} X$. By Corollary 3.3, the strategic-optimal classifier is parallel to the Non-strategic Bayes optimal classifier and has threshold $b_s^\star = b^\star + \Delta$ where $\Delta := \beta \max_{i\in[d]} \frac{w_i^\star}{\alpha_i}$. Thus we have

$$f^\star(x) = \mathbb{1}\{w^{\star\top} x \ge b^\star\} \qquad \text{and} \qquad f_s^\star(x) = \mathbb{1}\{w^{\star\top} x \ge b^\star + \Delta\}.$$

Note that for any classifier of the form $f_b(x) = \mathbb{1}\{w^{\star\top} x \ge b\}$, an agent with score $z = w^{\star\top} x$ can reach the acceptance region if and only if $z + \Delta \ge b$. Under the feasibility condition over $\mathcal{A}$ and the canonical minimum-cost positive best-response rule, an agent with original score $z = (w^\star)^\top x$ either remains at x if $z < b - \Delta$ or $z \ge b$, and otherwise moves to a minimum-cost accepted report in $\mathcal{A}$ with score exactly b. Therefore the improvement-aware error of f_b can be written as

$$\texttt{err}_{\text{Imp}}(f_b) = \int_{(-\infty, b-\Delta)} g(z)\, dP_Z(z) + \int_{[b-\Delta, b)} \big(1 - g(b)\big)\, dP_Z(z) + \int_{[b,\infty)} \big(1 - g(z)\big)\, dP_Z(z).$$

We have,

$$\begin{aligned}\texttt{err}_{\text{Imp}}(f_{b^\star}) - \texttt{err}_{\text{Imp}}(f_{b^\star+\Delta}) = &\int_{[b^\star-\Delta, b^\star)} \Big(1 - g(b^\star) - g(z)\Big)\, dP_Z(z) \\ &+ \int_{[b^\star, b^\star+\Delta)} \Big(g(b^\star + \Delta) - g(z)\Big)\, dP_Z(z).\end{aligned}$$

By Assumption 1, g is continuous and non-decreasing and $b^\star = \inf\{t : g(t) \ge 1/2\}$ is finite; hence $g(b^\star) = 1/2$. Also, since g is non-decreasing, for every $z < b^\star$, we have $g(z) \le g(b^\star) = 1/2$, and for every $z \in [b^\star, b^\star + \Delta)$, $g(z) \le g(b^\star + \Delta)$. Hence both integrands are nonnegative, and therefore $\texttt{err}_{\text{Imp}}(f_s^\star) \le \texttt{err}_{\text{Imp}}(f^\star)$. □

The Non-strategic Bayes optimal classifier, while not recognizing improvement, also ignores manipulation; representing a naive approach to classifier design, whereas an optimal strategic classifier treats manipulation as necessarily malicious and represents a more pessimistic approach. Theorem 3.4 shows that the pessimistic, strategic classifier $f_s^\star$ is a provably better no-post-deployment-label surrogate of an improvement aware classifier than the Bayes classifier.

As discussed earlier, if one needs to learn an optimal improvement aware classifier, in training, we need access to the post-deployment labels. The reason is that the labels are non-deterministically updated. Even if we assume an oracle access to post-deployment labels in a controlled or experimental deployment, the problem of learning an *optimal* improvement-aware classifier from non-manipulated data remains nontrivial. Next, we develop algorithm that operate under this stronger information model and addresses algorithmic and statistical challenges that arise even in this idealized setting.

4 Learnability Guarantee and Proposed Algorithm

Our first result establishes PAC learnability of the improvement-aware strategic classification problem by reducing it to a uniform convergence argument over a suitable hypothesis class. We assume oracle access to post-deployment labels; i.e., given f, the learner observes (x^f, y_{x^f}). Conditioned on the original sample points, these oracle labels are independent Bernoulli draws with mean $\eta(x^f)$.

Lemma 4.1. *Let $\mathcal{F}$ be the class of linear classifiers $f_{w,b}(x) = \mathbb{1}(w^\top x \ge x)$ with $w \in \mathbb{R}^d_{\ge 0}$, $w \ne 0$, $b \in \mathbb{R}$, under the conditions of Theorem 3.2. Under Assumption 1 , let $g^{-1}(u) := \inf\{t \in \mathbb{R} : g(t) \ge u\}$ denote the generalized inverse of the single-index link function. Define the induced oracle-loss class:*

$$H := \{(x, u) \mapsto \mathbb{1}[f(x^f) \ne \mathbb{1}\{u \le \eta(x^f)\}]\}$$

Then $d_\Delta := VC(H) = \mathcal{O}(d^2 \log(d))$.

Proof. Define $\mathcal{S} \coloneqq \{(x,u) \mapsto f(x^f) : f \in \mathcal{F}\}$, $\quad \mathcal{G} \coloneqq \{(x,u) \mapsto \mathbb{1}\{u \le \eta(x^f)\}\}$. Since each $f \in \mathcal{F}$ and $\mathbb{1}\{u \le \eta(x^f)\}$ are $\{0,1\}$-valued, the disagreement class H satisfies $H = \mathcal{S}\Delta\mathcal{G}$, where Δ denotes symmetric difference. We bound $\mathrm{VC}(\mathcal{S})$ and $\mathrm{VC}(\mathcal{G})$ separately.
Bounding $d_{\mathcal{S}} \coloneqq \mathrm{VC}(\mathcal{S})$. By Theorem 3.2, for $f = f_{w,b} \in \mathcal{F}$, $f(x^f) = \mathbb{1}\{w^\top x \ge b - \beta r(w)\}$, where $r(w) = \max_{i\in[d]} w_i/\alpha_i$. Hence $\mathcal{S}$ is a subclass of the unrestricted linear class $\{(x,u) \mapsto \mathbb{1}\{w^\top x \ge b'\} : w \in \mathbb{R}^d, b \in \mathbb{R}\}$, obtained by restricting w to $\mathbb{R}^d_{\ge 0}$. Since VC dimension of the unrestricted linear class is $d+1$, therefore we obtain $d_{\mathcal{S}} \le d+1$.
Bounding $d_{\mathcal{G}} \coloneqq \mathrm{VC}(\mathcal{G})$. By Assumption 1, we have $\eta(x) = g(w^{\star\top} x)$ for non-decreasing link g, $\mathbb{1}\{u \le \eta(x^f)\} = \mathbb{1}\{g^{-1}(u) \le w^{\star\top} x\}$. Under the non-trivial case $x^f \ne x$, Theorem 3.2 gives $x^f = x + \frac{b - w^\top x}{w_{i^\star}} e_{i^\star}$, where $i^\star \in \arg\max_{i\in[d]} w_i/\alpha_i$, so $w^{\star\top} x^f = w^{\star\top} x + \frac{w^\star_{i^\star}}{w_{i^\star}}(b - w^\top x)$. Evaluating this expression (w,b,x,u) requires: $\mathcal{O}(d)$ comparisons to determine $i^\star$; $\mathcal{O}(d)$ arithmetic operations to compute the dot product $w^\top x$ and $w^{\star\top} x$; $\mathcal{O}(1)$ further arithmetic and division operations to assemble the final value and compare it against $g^{-1}(u)$. Thus the computations uses $t = \mathcal{O}(d)$ operations of the type covered by Theorem 2.3 of Bartlett and Maass [7], over $d+1$ real parameters (w,b). Then, $d_{\mathcal{G}} = \mathcal{O}(d^2)$.
Combining via symmetric difference. Applying the standard VC bound for symmetric difference classes [31] to $H = \mathcal{S}\Delta\mathcal{G}$, $d_\Delta = \mathrm{VC}(H) = \mathcal{O}((d_{\mathcal{S}} + d_{\mathcal{G}})\log(d_{\mathcal{S}} + d_{\mathcal{G}})) = \mathcal{O}(d^2 \log d)$ $\square$

Theorem 4.2. *Fix a measurable tie-breaking rule so that x^f is well defined for every $x \in \mathcal{X}_0$ and $f \in \mathcal{F}$. Let $x_1, \dots, x_n \overset{\text{i.i.d.}}{\sim} \mathcal{D}$, draw independent $U_1, \dots, U_n \sim \mathrm{Unif}[0,1]$, and define the oracle label for classifier f by $y_{x_i^f} := \mathbf{1}\{U_i \le \eta(x_i^f)\}$; equivalently, for every fixed f, $y_{x_i^f} \sim \mathrm{Bernoulli}(\eta(x_i^f))$. Define $\widehat{err}_{\mathrm{Imp}}(f) := \frac{1}{n}\sum_{i=1}^n \mathbf{1}[f(x_i^f) \ne y_{x_i^f}]$. From Lemma 4.1 , induced oracle-loss class $H := \{(x,u) \mapsto \mathbf{1}[f(x^f) \ne \mathbf{1}\{u \le \eta(x^f)\}] : f \in \mathcal{F}\}$ has finite VC dimension d_Δ. Then, for every $\delta > 0$, with probability at least $1-\delta$ over the draw of $(x_i, U_i)_{i=1}^n$,*

$$\sup_{f\in\mathcal{F}} |err_{\mathrm{Imp}}(f) - \widehat{err}_{\mathrm{Imp}}(f)| \le \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d_\Delta \log(en/d_\Delta)}{n}} + \sqrt{\frac{2\log(1/\delta)}{n}}.$$

Remark 4.3. Theorem 4.2 analyzes the sampled-oracle setting. The auxiliary variables U_i provide a common coupling of the post-response labels across classifiers: for every fixed f, $y_{x_i^f} = \mathbf{1}\{U_i \le \eta(x_i^f)\}$ has distribution $\mathrm{Bernoulli}(\eta(x_i^f))$. This lets us view the empirical improvement-aware error as the empirical mean of a binary loss class over the augmented sample (x_i, U_i). In the algorithmic section, when η is replaced by an estimator $\widehat{\eta}$, the resulting plug-in analysis incurs an additional approximation term of order $\sup_{z\in\mathcal{A}} |\widehat{\eta}(z) - \eta(z)|$.

The main statistical difficulty is that the post-response label $y_{x_i^f}$ changes with f. To obtain a uniform convergence statement over classifiers, we couple these labels using a shared auxiliary random variable U_i for each original sample x_i. This makes the loss induced by every classifier a function of the same augmented sample (x_i, U_i). This construction allows us to analyze the induced oracle-loss class

$$H = \{(x,u) \mapsto \mathbf{1}[f(x^f) \ne \mathbf{1}\{u \le \eta(x^f)\}] : f \in \mathcal{F}\},$$

rather than only the strategic disagreement class. The resulting bound depends on the VC dimension of this induced binary class. The proof of Theorem 4.2 is given in Appendix D.

4.1 Proposed Algorithm: STRAT-IMP-AWARE

We now propose an improvement-aware strategic learning algorithm that minimizes classification error under endogenous feature manipulation and label improvement. The proposed algorithm extends ideas from Levanon and Rosenfeld [27] in three ways: (i) decomposable costs replace L_2 costs, yielding a closed-form strategic shift $\beta \cdot \max_{j\in[d]} w_j/\alpha_j$; (ii) an improvement-aware oracle simulates endogenous label flips under the post-deployment distribution; and (iii) the training loop jointly optimizes over strategic shifts and oracle-adjusted labels.

The proposed algorithm STRAT-IMP-AWARE, simulates post-deployment behavior during training. Given $\{(x_i, y_i)_{i=1}^n\}$ drawn from the pre-deployment distribution, each x_i is mapped to its strategic best response x_i^f, which under non-negative weights and decomposable costs is a non-decreasing feature transformation moving the point toward the decision boundary with minimal effort; already positive points satisfy $x_i^f = x_i$. Since η is unknown in practice, STRAT-IMP-AWARE uses a plug-in estimator $\widehat{\eta}$ and samples an improvement-aware label $y_i^{\text{Imp}} \sim \text{Bernoulli}(\widehat{\eta}(x_i^f))$. Training then minimizes a hinge-type surrogate with the closed-form shifted margin $\beta \max_{j\in[d]} w_j/\alpha_j$. Optimization proceeds via SGD, alternating between simulating strategic responses, sampling oracle labels, and updating classifier parameters.

Our next result (Theorem 4.4) gives a learnability guarantee of STRAT-IMP-AWARE (Algorithm 3) under sampled oracle labels. The algorithm below is more practical: it replaces the unknown outcome law η by an independently trained estimator $\widehat{\eta}$, simulates post-response labels using $\widehat{\eta}$, and optimizes a strategic hinge surrogate. Theorem 4.4 separates the resulting error into a generalization term and a plug-in estimation term. Let $\rho_{\text{Imp}} := r + \frac{\beta}{\min_{j\in[d]} \alpha_j}$, $\varepsilon_\eta := \sup_{z\in\mathcal{A}} |\widehat{\eta}(z) - \eta(z)|$ and $\widehat{L}^{\widehat{\eta}}_{\text{hinge}}(w, b)$ denote the empirical plug-in strategic hinge loss minimized by STRAT-IMP-AWARE.

Theorem 4.4. *Suppose Assumptions 1 and 2 hold, $\mathcal{X}_0$ is bounded with $r := \sup_{x\in\mathcal{X}_0} \|x\|_2 < \infty$, and $\widehat{\eta}$ is estimated independently of the training sample. Assume* STRAT-IMP-AWARE *uses the plug-in oracle $\widehat{\eta}$, a fixed measurable best-response rule, and returns $(\widehat{w}, \widehat{b})$ with $\|\widehat{w}\|_2 \le k$ and $|\widehat{b}| \le B$. Then, for every $\delta \in (0, 1)$, with probability at least $1 - \delta$,*

$$\mathit{err}_{\text{Imp}}(f_{\widehat{w},\widehat{b}}) \le \widehat{L}^{\widehat{\eta}}_{\text{hinge}}(\widehat{w}, \widehat{b}) + \frac{2(k\rho_{\text{Imp}} + B)}{\sqrt{n}} + (1 + k\rho_{\text{Imp}} + B)\sqrt{\frac{2\log(2/\delta)}{n}} + 2(k\rho_{\text{Imp}} + B)\varepsilon_\eta.$$

The proof of Theorem 4.4, deferred to Appendix E, applies uniform convergence to the plug-in strategic hinge-loss class over the bounded parameter set $\mathcal{W}_{k,B}$. Decomposable costs yield the closed-form strategic shift $S(w) = \beta \max_{j\in[d]} w_j/\alpha_j$, and the boundedness assumptions give the margin bound $C_{k,B} = k\rho_{\text{Imp}} + B$. Combining the resulting Rademacher bound with standard bounded-loss generalization and the plug-in error ε_η gives the stated guarantee.

Remark 4.5. The term $\varepsilon_\eta = \sup_{z\in\mathcal{A}} |\widehat{\eta}(z) - \eta(z)|$ separates the statistical error of estimating the class-probability function from the generalization error of the strategic hinge class. In Appendix F, we show that under a correctly specified single-index model with a Lipschitz link and a strongly convex population risk, a calibrated plug-in estimator satisfies $\varepsilon_\eta = \mathcal{O}(\sqrt{(d + \log(1/\delta))/m})$ with high probability, where m is the sample size used to estimate $\widehat{\eta}$.

Algorithm 1 ORACLE

Input: Sample (x, y), Model (w, b), Cost-coeff. $\vec{\alpha}$, estimate $\widehat{\eta}$

$$\text{BR}(x; w, b) \leftarrow \arg\max_{z\in\mathcal{A}} \Big(\beta\, \mathbb{1}(w^\top z \ge b) - \textstyle\sum_j \alpha_j (z_j - x_j)_+\Big)$$

Sample $y_{\text{Imp}} \sim \text{Bernoulli}(\widehat{\eta}(\text{BR}(x; w, b)))$.

return $\tilde{y} \leftarrow 2y_{\text{Imp}} - 1$

Algorithm 2 STRAT-IMP-ERROR

Input: Feature x, Target $\tilde{y}$, Model (w, b), Utility $\beta > 0$, Cost-coeff. $\vec{\alpha}$

pred $\leftarrow w^\top x - b$

$S(w) \leftarrow \beta \max_{j\in[d]} \left(\frac{w_j}{\alpha_j}\right)$

Margin $\leftarrow \tilde{y} \cdot (\text{pred} + S(w))$

return $\mathcal{L} \leftarrow \max(0, 1 - \text{Margin})$

Algorithm 3 STRAT-IMP-AWARE

Input: Training Set $\mathcal{D}_{tr}$, Validation Set $\mathcal{D}_{val}$, Model (w, b), Epochs T, Learning rate γ, Utility β, Cost-coeff. $\vec{\alpha}$, estimate $\widehat{\eta}$

Initialize w, b, $L_{best} \leftarrow \infty$

for $epoch = 1 \dots T$ **do**

  for batch $(X, Y) \in \mathcal{D}_{tr}$ **do**

    $(X, \tilde{Y}) \leftarrow \textsc{Oracle}(X, Y, w, b, \vec{\alpha}, \widehat{\eta})$

    $\mathcal{L}_{batch} \leftarrow \textsc{StratImpError}(X, \tilde{Y}, w, b, \beta, \vec{\alpha})$

    $w \leftarrow w - \gamma \nabla_w \mathcal{L}_{batch}$

    $b \leftarrow b - \gamma \nabla_b \mathcal{L}_{batch}$

  end for

  $L_{val} \leftarrow 0$

  for batch $(X, Y) \in \mathcal{D}_{val}$ **do**

    $(X, \tilde{Y}) \leftarrow \textsc{Oracle}(X, Y, w, b, \vec{\alpha})$

    $L_{val} \leftarrow L_{val} + \textsc{StratImpError}(X, \tilde{Y}, w, b, \beta, \vec{\alpha})$

  end for

  $L_{val} \leftarrow \text{mean}(L_{val})$

  if $L_{val} < L_{best}$ **then**

    $L_{best} \leftarrow L_{val}$, $w^\star \leftarrow w$, $b^\star \leftarrow b$

  end if

end for

return Best Model $(w^\star, b^\star)$

5 Experiments

This section and in Appendix G, we present a comprehensive set of simulation and empirical experiments designed to validate the theoretical assumptions of our framework and the practical performance of our proposed Strategic-Improvement-aware algorithm. Our code is publically available here.

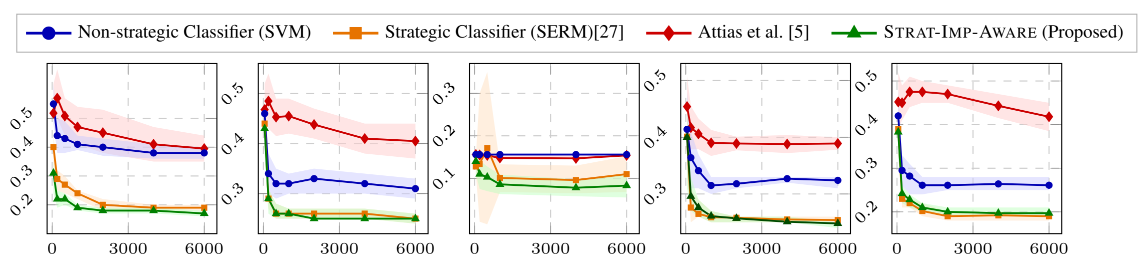
Experimental Setting: We evaluate our algorithm on four real-world datasets (Adult Income [9], HELOC [10], Law School [26] and ACS Income [14]) and one synthetic dataset. We perform two major experiments on these datasets. Additional details on experimental setup with ablation studies are given in the Appendix G. The description of the datasets is given in the Table 1. The class-probability function $\eta(x)$ is modeled using Logistic Regression with the Calibrated classification technique, following the Assumption 1. On all the datasets, we consider the features for which $\eta(x)$ is monotonically increasing, more details can be found in Appendix G. We use a single-layer neural network with a d-dimensional input and a biased linear output and optimize the model parameters using a Hinge loss objective. The model is trained using stochastic gradient descent (SGD).

We implement two benchmark algorithms namely SERM ([27]) and [5] and the improvement-aware classifier STRAT-IMP-AWARE. The classifier is trained using an objective that anticipates agent manipulation. Specifically, the model identifies the feature with the highest return on investment calculated as the ratio of the model weight to the manipulation cost (w_k/α_k). During training, we adjust the loss by adding a strategic gain term, which represents the maximum score change an agent can achieve within their budget β.

Dataset	**Instances**	**Total features**	**Imp. features**	**Task**
Adult	48,842	15	5	Income $>$ 50K
HELOC	10,459	24	8	Credit risk
Law School	20,798	12	6	Bar exam
ACS Income	1,664,500	10	3	Income $>$ 50K
Synthetic	20,000	8	8	+ve label

Table 1: Datasets used in experiments.

Robustness to Training Set Size. Figure 1 evaluates the improvement-aware strategic error across the datasets in Table 1 as a function of the number of training samples and shows that the improvement-aware linear classifier $f^\star_{\text{Imp}}$ consistently achieves lowest improvement error regardless of training set size. Furthermore, the strategic linear classifier $f^\star_s$ generally outperforms the standard optimal linear classifier $f^\star$, while achieving comparable performance on the Law School dataset. This clear hierarchy empirically demonstrates that $f^\star_s$ serves as a better proxy for improvement-aware settings than the naive $f^\star$, even when training data is limited.



(a) Adult (b) HELOC (c) Law School (d) ACSIncome (e) Synthetic

Figure 1: **Improvement-aware strategic error versus training sample size across five datasets.** We compare four linear classifiers: a non-strategic baseline (SVM), a strategic classifier [27], the method of Attias et al. [5], and our proposed STRAT-IMP-AWARE algorithm. Shaded regions denote the min–max range over 10 runs.

5.1 Experimental findings

Test-Time Improvement Rates. We evaluate the agent's label improvement across the four classifiers:SVM, SERM [27], STRAT-IMP-AWARE and Attias et al. [5]. We split each dataset 70-30 among training and test sets. For each deployed classifier, we count an agent as manipulating if its selected best response differs from its original feature vector. Among manipulating agents, we count an agent as improved if the sampled post-response label is positive after manipulation when the original label was negative. The improvement rate reported in 2 is the fraction of manipulating agents who improve. All models are trained on the complete training set, while the total counts of manipulating and improved agents are quantified using the test data.

Dataset	Metric	SVM	SERM [27]	STRAT-IMP AWARE	Attias et al. [5]
Adult	Manip. agents	12169	775	**13464**	4480
	Improved agents	3567	90	**9623**	2858
	Imp Rate (%) ↑	29.31	11.61	**71.47**	63.79
HELOC	Manip. agents	1761	1048	**3138**	1014
	Improved agents	609	220	**1572**	498
	Imp Rate (%) ↑	34.58	20.99	**50.10**	49.11
Law School	Manip. agents	33	22	32	**77**
	Improved agents	9	4	11	**33**
	Imp Rate (%) ↑	27.27	18.18	34.37	**42.86**
ACS Income	Manip. agents	32369	17303	**52666**	21062
	Improved agents	12388	4163	**26835**	10184
	Imp Rate (%) ↑	38.27	24.05	**50.95**	48.35
Synthetic	Manip. agents	3027	2233	**5992**	2779
	Improved agents	1057	232	**2968**	1321
	Imp Rate (%) ↑	34.91	10.38	**49.53**	47.54

Table 2: Table shows that STRAT-IMP-AWARE attains the highest improvement rate on four of five datasets while also inducing more manipulation. The only exception is Law School, likely due to the small number of manipulating agents and class imbalance.

6 Discussion

We studied a strategic classification setting in which deployment-time responses can change both observable features and true qualifications. The main takeaway is not that strategic conservatism is always optimal, but that when feature changes can also improve true outcomes, ignoring strategic response can be worse than using a conservative strategic surrogate. In particular, under our structural assumptions, the strategic-optimal classifier is a closer proxy for the improvement-aware objective than the Bayes classifier when post-deployment labels are unavailable. We also established PAC-style learnability with oracle access to post-response labels and proposed a plug-in learning algorithm with generalization guarantees.

Limitations. Our analysis relies on a single-index qualification model, linear classifiers, linear-decomposable costs, and a canonical best-response rule. These assumptions make the problem tractable but may limit applicability in richer real-world settings. Important directions include extending the analysis to non-separable costs such as ℓ_2 costs, heterogeneous or repeated-interaction models, cost-sensitive objectives, and causal models of improvement. Another useful direction is to derive sharper quantitative comparisons between Bayes, strategic, and improvement-aware classifiers in terms of distributional properties such as mass near the decision boundary, curvature of the class-probability function, and heterogeneity in effort costs.

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A Notation

Table 3: Notation used in the paper.

Notation	Description
$\mathcal{X} \subseteq \mathbb{R}^d$	Valid feature space; both original features and reported/post-response features lie in $\mathcal{X}$.
$\mathcal{X}_0 = \mathrm{supp}(D) \subseteq \mathcal{X}$	Support of the original/pre-deployment feature distribution.
$\mathcal{A} \subseteq \mathcal{X}$	Feasible report/action space, with $\mathcal{X}_0 \subseteq \mathcal{A}$.
D	Distribution over original features, supported on $\mathcal{X}_0$.
$x \in \mathcal{X}_0$	Original feature vector sampled from D.
$z \in \mathcal{A}$	Reported/post-response feature vector.
$Y = \{0, 1\}$	Binary label space.
$\eta : \mathcal{X} \to [0, 1]$	Class-probability function, $\eta(x) = \Pr(Y = 1 \mid X = x)$.
$y_x \sim \mathrm{Bernoulli}(\eta(x))$	Conditional label generated at feature vector $x \in \mathcal{X}$.
$\mathcal{F}$	Hypothesis class of linear classifiers $f_{w,b} : \mathcal{X} \to \{0, 1\}$.
$f_{w,b}(z) = \mathbb{1}\{w^\top z \geq b\}$	Linear classifier with weight vector $w \in \mathbb{R}^d_{\geq 0}$ and threshold $b \in \mathbb{R}$.
$c : \mathcal{X}_0 \times \mathcal{A} \to \mathbb{R}_+$	Cost of changing original feature $x \in \mathcal{X}_0$ to report $z \in \mathcal{A}$.
$c(x, z) = \sum_{i=1}^d \alpha_i (z_i - x_i)_+$	Linear-decomposable one-sided cost, with $\alpha_i > 0$.
$(a)_+ = \max\{a, 0\}$	Positive part of $a \in \mathbb{R}$.
$\beta > 0$	Utility gain from receiving positive classification.
$u_f(x, z) = \beta f(z) - c(x, z)$	Utility of reporting $z \in \mathcal{A}$ for an agent with original feature $x \in \mathcal{X}_0$.
$x^f \in \arg\max_{z \in \mathcal{A}} u_f(x, z)$	Selected best response/report under classifier f.
$\mathrm{BR}(x; w, b)$	Best response of x to the linear classifier $f_{w,b}$.
$\mathtt{err}(f) = \mathbb{E}_{x \sim D}[\mathbb{1}\{f(x) \neq y_x\}]$	Standard non-strategic classification error.
$\mathtt{err}_s(f) = \mathbb{E}_{x \sim D}[\mathbb{1}\{f(x^f) \neq y_x\}]$	Strategic error with immutable labels.
$\mathtt{err}_{\mathrm{Imp}}(f) = \mathbb{E}_{x \sim D}[\mathbb{1}\{f(x^f) \neq y_{x^f}\}]$	Improvement-aware strategic error with post-response labels.
$\widehat{\mathtt{err}}_{\mathrm{Imp}}(f) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{f(x_i^f) \neq y_{x_i^f}\}$	Empirical improvement-aware strategic error.
$g : \mathbb{R} \to [0, 1]$	Monotone link function in the single-index model.
$w^\star \in \mathbb{R}^d_{\geq 0}$	True single-index direction, satisfying $\eta(x) = g((w^\star)^\top x)$.
$b^\star = \inf\{t : g(t) \geq 1/2\}$	Bayes threshold under the single-index model.
$f^\star$	Non-strategic Bayes optimal classifier.
$f_s^\star$	Strategic-optimal shifted classifier.
$f_{\mathrm{Imp}}^\star$	Improvement-aware optimal classifier.
$r(w) = \max_{i \in [d]} w_i / \alpha_i$	Cost-adjusted maximum coordinate ratio.
$S(w) = \beta \max_{i \in [d]} w_i / \alpha_i$	Strategic shift induced by classifier $f_{w,b}$.
$b_s = b + S(w)$	Shifted threshold corresponding to $f_{w,b}$.
$M_{w,b}(x) = w^\top x - b + S(w)$	Strategic shifted margin used in the hinge surrogate.
$\rho_{\mathrm{Imp}} = r + \beta / \min_{j \in [d]} \alpha_j$	Radius parameter used in the generalization bound, where $r = \sup_{x \in \mathcal{X}_0} \lVert x \rVert_2$.
$\widehat{\eta}$	Plug-in estimator of η.
$\varepsilon_\eta = \sup_{z \in \mathcal{A}} \lvert \widehat{\eta}(z) - \eta(z) \rvert$	Uniform plug-in estimation error over feasible reports.
$\widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(w, b)$	Empirical plug-in strategic hinge loss minimized by STRAT-IMP-AWARE.
$\mathcal{W}_{k,B}$	Norm- and bias-bounded parameter class $\{(w, b) : w \in \mathbb{R}^d_{\geq 0}, \lVert w \rVert_2 \leq k, \lvert b \rvert \leq B\}$.
$C_{k,B} = k\rho_{\mathrm{Imp}} + B$	Uniform margin bound used in Theorem 4.4.

B Additional Proofs from Section 2

Observation 2.4. *[Existence of a linear Non-strategic Bayes optimal classifier] Under* 0–1 *loss in the Single-index qualification model (Assumption 1), there exists a Non-strategic Bayes optimal classifier of the form* $f^\star(x) = \mathbf{1}\{{w^\star}^\top x \geq b^\star\}$ *for some threshold* $b^\star \in \mathbb{R}$*. In particular, if* $b^\star := \inf\{t \in \mathbb{R} : g(t) \geq 1/2\}$*, then the classifier above is Bayes-optimal.*

Proof. By Assumption 1,

$$\eta(x) = g(w^{\star\top} x).$$

Since g is nondecreasing, the set

$$\{x : \eta(x) \geq 1/2\} = \{x : g(w^{\star\top} x) \geq 1/2\}$$

is a threshold set of the form $\{x : w^{\star\top} x \geq b^\star\}$, where

$$b^\star := \inf\{t \in \mathbb{R} : g(t) \geq 1/2\}.$$

Hence the classifier $f^\star(x) = \mathbf{1}\{w^{\star\top} x \geq b^\star\}$ is Bayes optimal. $\square$

C Additional Proofs from Section 3

In this section, we present proofs from Section 3.

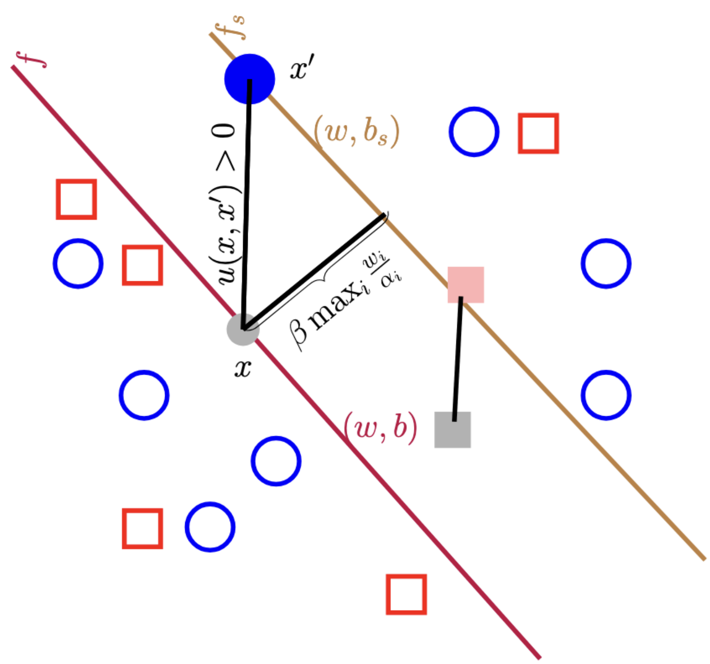


Figure 2: The strategic classifier f_s (brown) is obtained by shifting the linear classifier f (red) by a margin of $\max_{i\in[d]} \left(\frac{w_i}{\alpha_i}\right) \beta$. This translation anticipates utility-maximizing feature manipulation from x to x' by the agents.

Lemma 3.1. *Suppose $\mathcal{A}$ is coordinate-wise order-convex: for every $x \in \mathcal{X}_0$, $z \in \mathcal{A}$, and every $\tilde{z}$ lying coordinate-wise between x and z, we have $\tilde{z} \in \mathcal{A}$. Let $f_{w,b} \in \mathcal{F}$ be a linear classifier with $w \in \mathbb{R}^d_{\geq 0}$ and $b \in \mathbb{R}$. Then, for every $x \in \mathcal{X}_0$, there exists a strategic best response $x^f \in \mathcal{A}$ satisfying $x_i^f \geq x_i$ for all $i \in [d]$.*

Proof. We prove the lemma by contradiction. Suppose there exists some $i \in [d]$ such that $x_i^f < x_i$. We will show that there exists an alternative manipulated point $\tilde{x}$ such that $\tilde{x}_i = x_i$ and show that $u_f(x, \tilde{x}) \geq u_f(x, x^f)$, thereby allowing us to restrict our attention to manipulated points that are coordinate-wise non decreasing.

We analyze the properties of the manipulated point x^f by distinguishing cases based on the relative positions of x and x^f with respect to decision boundary.

Case 1: $w^\top x \geq b$**.** In this case, the agent is already classified positively. Then, the optimal strategy is $x^f = x$ (zero cost). Hence, the condition holds trivially.

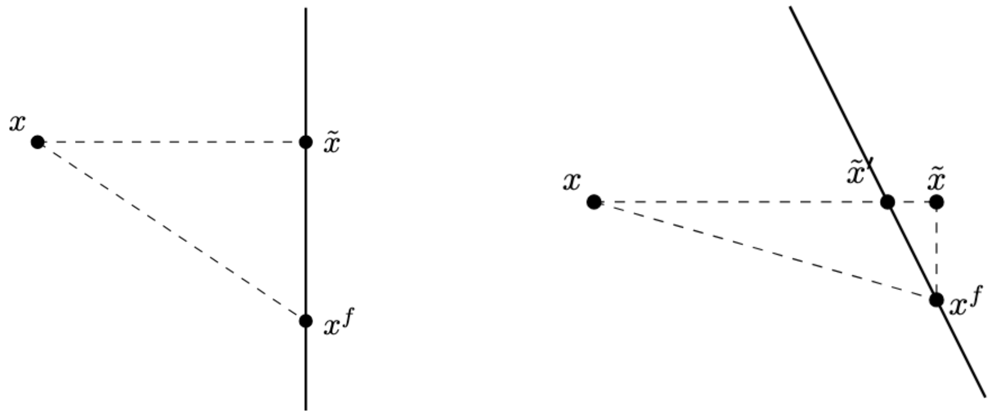


Figure 3: Strategic response under feature manipulation constraints. **Left:** There exists a coordinate $i \in [d]$ such that $w_i = 0$. In this case, the cost of moving from x to $\tilde{x}$ is equal to the cost of moving to any other feasible point (e.g., x^f), since changes along this feature do not affect the prediction. Hence, the agent can manipulate x to $\tilde{x}$ without incurring additional cost. **Right:** For all $i \in [d]$, $w_i > 0$. Under the cost structure, the cost of moving from x to x^f is equal to the cost of moving from x to $\tilde{x}$. Moreover, the cost of moving from x to $\tilde{x}$ is strictly greater than the cost of moving from x to $\tilde{x}'$. Therefore, any point x will optimally manipulate to $\tilde{x}'$ rather than $\tilde{x}$.

Case 2: $w^\top x < b$ and $w^\top x^f < b$. The agent fails to cross the classifier boundary; again, $x^f = x$.

Case 3: $w^\top x < b$ and $w^\top x^f \geq b$. Since x^f is a strategic manipulation, it must lie on the decision boundary to minimize cost, so $w^\top x^f = b$. Let $J = \{i \in [d] \mid x_i^f < x_i\}$ with $J \neq \emptyset$. Consider an index $i \in J$. We construct an intermediate point $\tilde{x}$ (see Figure 3 for reference) such that:

$$\tilde{x}_j = \begin{cases} x_i & \text{if } j = i, \\ x_j^f & \text{if } j \neq i \end{cases}$$

Since $x_i^f < x_i$, the term $|x_i^f - x_i|_+$ in the cost function is 0. Similarly, for $\tilde{x}$, $|\tilde{x}_i - x_i|_+ = 0$. Thus, the cost remains unchanged:

$$c(x, x^f) = \sum_{j=1}^{d} \alpha_j |x_j^f - x_j|_+ = \sum_{j \neq i} \alpha_j |x_j^f - x_j|_+ = c(x, \tilde{x})$$

Furthermore, we analyze the score of $\tilde{x}$:

$$w^\top \tilde{x} = w^\top x^f + w_i(x_i - x_i^f) = b \, + \, w_i(x_i - x_i^f)$$

Since $w_i \geq 0$, and $x_i > x_i^f$, let $\epsilon = w_i(x_i - x_i^f) \geq 0$.

Sub-case 3.1: $w_i = 0$ If $w_i = 0$, then $\epsilon = 0$ and $w^\top \tilde{x} = w^\top x^f = b$. Since both $\tilde{x}$ and x^f lie on the decision boundary (see figure 3) and have equal costs, $\tilde{x}$ is also a valid maximizer of the utility. We can repeat this replacement for all $i \in J$ until all components satisfy the condition without loss of generality.

Sub-case 3.2: $w_i > 0$ In this scenario, the intermediate point $\tilde{x}$ satisfies:

$$w^\top \tilde{x} = w^\top x^f + w_i(x_i - x_i^f) = b + \epsilon, \quad \text{Where } \epsilon > 0$$

We also established previously that $c(x, \tilde{x}) = c(x, x^f)$. We observe that because $w^\top x^f = b > w^\top x$ and $w \geq 0$, there must exist at least one index $k \in [d]$ such that $w_k > 0$ and $x_k^f > x_k$. (if no such k existed, it would imply $x_j^f \leq x_j$ for all j with non-negative weights, leading to $w^\top x^f \leq w^\top x < b$, a contradiction).

We construct a new point $\tilde{x}'$ by reducing the k-th component of $\tilde{x}$ to satisfy the boundary constraint (see figure 3). Let $\gamma = \epsilon / w_k > 0$. We define $\tilde{x}'$ as:

$$\tilde{x}'_j = \begin{cases} \tilde{x}_k - \gamma & \text{if } j = k, \\ \tilde{x}_k & \text{if } j \neq k \end{cases}$$

First, we verify the boundary condition; $w^\top \tilde{x}' = w^\top \tilde{x} - w_k \gamma = (b + \epsilon) - w_k \left(\frac{\epsilon}{w_k}\right) = b$.

Second, we analyze the cost. Since $\tilde{x}_k = x^f_k > x_k$, we observe that:

$$\begin{aligned} c(x, x^f) = c(x, \tilde{x}) &= \sum_{j=1}^{d} \alpha_j |\tilde{x}_j - x_j|_+ \\ &= \sum_{j \neq k} \alpha_j |\tilde{x}'_j - x_j|_+ \; + \; \alpha_k |\tilde{x}_k - x_k|_+ \\ &= \sum_{j \neq k} \alpha_j |\tilde{x}'_j - x_j|_+ \; + \; \alpha_k \, |\tilde{x}'_k - x_k + \gamma_k|_+ \\ &= \sum_{j \neq k} \alpha_j |\tilde{x}'_j - x_j|_+ \; + \; \alpha_k |\tilde{x}'_k - x_k|_+ + \alpha_k \gamma_k \qquad \{\because \alpha_k, \gamma_k > 0\} \\ &= c(x, \tilde{x}') + \alpha_k \gamma_k > c(x, \tilde{x}') \end{aligned}$$

Thus, we have found a point $\tilde{x}'$ on the decision boundary ($w^\top \tilde{x}' = b$) with strictly lower cost than x^f. This contradicts the assumption that x^f is the strategic response.

□

Corollary 3.3. *Suppose the assumptions of Theorem 3.2 hold for every linear classifier in $\mathcal{F}$, including the feasibility and tie-breaking assumptions. Let $f^\star$ denote the Non-strategic Bayes optimal linear classifier with parameters $(w^\star, b^\star)$. Then the shifted classifier $f^\star_s(x) = \mathbf{1}\{(w^\star)^\top x \geq b^\star + \beta \max_i w^\star_i / \alpha_i\}$ is strategic-optimal among linear classifiers.*

Proof. Let $f^\star$ be the Non-strategic Bayes optimal linear classifier and $f^\star_s$ be the corresponding strategic classifier satisfying $\texttt{err}_s(f^\star_s) = \texttt{err}(f^\star)$. For contradiction, assume that there is a classifier $f^{'}_s$ with parameters (w', b') satisfying

$$\texttt{err}_s(f^{'}_s) < \texttt{err}_s(f^\star_s). \tag{4}$$

and consider classifier $f^{'}$ with parameters $w', b^{'} - \max_{i \in [d]} \left(\frac{w^{'}_i}{\alpha_i}\right) \beta$. The following set of inequalities result in a contradiction.

$$\texttt{err}_s(f^\star_s) = \texttt{err}(f^\star) \leq \texttt{err}(f^{'}) = \texttt{err}_s(f^{'}_s) < \texttt{err}_s(f^*_s).$$

The first inequality above follows from Bayes optimality of $f^\star$ and the second inequality follows from Equation 4. □

Lemma C.1. *[Improvement threshold lies between Non-strategic Bayes and Strategic thresholds] Let $x \sim \mathcal{D}$ with $x \in \mathcal{X} \subseteq \mathbb{R}$, and the label associated with input x be $y_x \sim \mathrm{Bernoulli}(\eta(x))$ where $\eta : \mathbb{R} \to [0, 1]$ is an increasing function. Consider threshold classifiers of the form $f_\theta(x) = \mathbb{1}(x \geq \theta)$. Denote by $\theta^\star$ the Non-strategic Bayes optimal threshold. Let θ_s be the optimal threshold under strategic manipulation with $\exists \alpha \in \mathbb{R}^+$ then suppose cost function $c(x, x') = max(0, \alpha(x' - x))$ and let θ_{imp} be the optimal threshold under our improvement-aware model. Then the thresholds satisfy:*

$$\theta^\star \;\; \leq \;\; \theta_{\mathrm{imp}} \;\; \leq \;\; \theta_s.$$

Proof. We establish the inequality chain $\theta^\star \leq \theta_{\mathrm{Imp}} \leq \theta_s$ via proof by contradiction. We rely on the monotonicity of the class posterior $\eta(x)$ under the generalized linear model assumption. We proceed in two parts, First, suppose that $\theta_{\mathrm{imp}} > \theta_s$. We will show that this violates the optimality condition of the improvement-aware classifier. Next, suppose that $\theta_{\mathrm{imp}} < \theta^\star$. We will again reach a contradiction

by showing that the improvement risk can be strictly reduced by raising the threshold. For simplicity in notation, let $x' = x^{f_{\theta_{\text{Imp}}}}$ be strategic manipulated point under $f_{\theta_{\text{Imp}}}$ threshold classifier.
We know that,

$$\begin{aligned}\theta_{\text{Imp}} &= \arg\min_{\theta\in\mathbb{R}} \texttt{err}_{\text{Imp}}(f_\theta)\\ &= \arg\min_{\theta\in\mathbb{R}} \mathbb{E}[\mathbb{1}(f_\theta(x^{f_\theta}) \neq y_{x^{f_\theta}})]\end{aligned}$$

We begin with a few supporting technical results.

Claim C.2.

$$\left\{\begin{aligned}&\mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))-\mathbb{P}(y_{\theta^\star}=0)(F(\theta^\star)-F(\theta^\star-\frac{\beta}{\alpha}))\\ &\quad+\int_{\theta_{\text{Imp}}}^{\theta^\star}\mathbb{P}(y_x=0)\,p_X(x)dx\,-\int_{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}^{\theta^\star-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\end{aligned}\right\}>0$$

Proof.

$$\begin{aligned}&\mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))-\mathbb{P}(y_{\theta^\star}=0)(F(\theta^\star)-F(\theta^\star-\frac{\beta}{\alpha}))\\ &\quad+\int_{\theta_{\text{Imp}}}^{\theta^\star}\mathbb{P}(y_x=0)\,p_X(x)dx\,-\int_{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}^{\theta^\star-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\ &\geq\ \mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))+\mathbb{P}(y_{\theta^\star}=0)(F(\theta^\star-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}))\\ &\quad-\int_{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}^{\theta^\star-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\ &\qquad\qquad\qquad\qquad(\because \forall x\in[\theta_{\text{Imp}},\theta^\star],\quad \mathbb{P}(y_x=0)\leq\mathbb{P}(y_{\theta_{\text{Imp}}}=0))\\ &>\mathbb{P}(y_{\theta^\star}=0)\,(F(\theta^\star-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))-\int_{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}^{\theta^\star-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\ &\qquad\qquad\qquad\qquad(\because \theta_{\text{Imp}}<\theta^\star\implies\mathbb{P}(y_{\theta_{\text{Imp}}}=0)>\mathbb{P}(y_{\theta^\star}=0))\\ &=\ (F(\theta^\star-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))(\mathbb{P}(y_{\theta^\star}=0)-1)+\int_{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}^{\theta^\star-\frac{\beta}{\alpha}}\mathbb{P}(y_x=0)\,p_X(x)dx\\ &\geq(F(\theta^\star-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))(\mathbb{P}(y_{\theta^\star}=0)+\mathbb{P}(y_{\theta^\star-\frac{\beta}{\alpha}}=0)-1)\\ &\qquad\qquad\qquad\qquad(\because \forall x\in[\theta_{\text{Imp}}-\tfrac{\beta}{\alpha},\theta^\star-\tfrac{\beta}{\alpha}],\quad\mathbb{P}(y_x=0)\geq\mathbb{P}(y_{\theta^\star-\frac{\beta}{\alpha}}=0))\\ &>(F(\theta^\star-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))(2\,\mathbb{P}(y_{\theta^\star}=0)-1)\\ &=0\qquad\qquad\qquad\qquad(\because\mathbb{P}(y_{\theta^\star}=0)=\tfrac{1}{2})\end{aligned}$$

Hence proved. □

Claim C.3.

$$\left\{\begin{aligned}&\mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))-\mathbb{P}(y_{\theta_s}=0)(F(\theta_s)-F(\theta_s-\frac{\beta}{\alpha}))\\ &\quad-\int_{\theta_s}^{\theta_{\text{Imp}}}\mathbb{P}(y_x=0)\,p_X(x)dx\,+\int_{\theta_s-\frac{\beta}{\alpha}}^{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\end{aligned}\right\}>0$$

Proof.

$$
\begin{aligned}
&\mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))-\mathbb{P}(y_{\theta_s}=0)(F(\theta_s)-F(\theta_s-\frac{\beta}{\alpha}))\\
&\quad-\int_{\theta_s}^{\theta_{\text{Imp}}}\mathbb{P}(y_x=0)\,p_X(x)dx\;+\int_{\theta_s-\frac{\beta}{\alpha}}^{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\
&\geq \mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))\;+\;\mathbb{P}(y_{\theta_s}=0)(F(\theta_s-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}))\\
&\quad+\int_{\theta_s-\frac{\beta}{\alpha}}^{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\
&\hspace{18em}(\because \forall x\in[\theta_s,\theta_{\text{Imp}}],\quad -\mathbb{P}(y_x=0)\geq-\mathbb{P}(y_{\theta_s}=0))\\
&> \mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_s-\frac{\beta}{\alpha})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))+\int_{\theta_s-\frac{\beta}{\alpha}}^{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\
&\hspace{20em}(\because \theta_s<\theta_{\text{Imp}}\implies \mathbb{P}(y_{\theta_s}=0)>\mathbb{P}(y_{\theta_{\text{Imp}}}=0))\\
&= (F(\theta_{\text{Imp}}-\frac{\beta}{\alpha})-F(\theta_s-\frac{\beta}{\alpha}))(1-\mathbb{P}(y_{\theta_{\text{Imp}}}=0))-\int_{\theta_s-\frac{\beta}{\alpha}}^{\theta_{\text{Imp}}-\frac{\beta}{\alpha}}\mathbb{P}(y_x=0)\,p_X(x)dx\\
&\geq (F(\theta_{\text{Imp}}-\frac{\beta}{\alpha})-F(\theta_s-\frac{\beta}{\alpha}))(1-\mathbb{P}(y_{\theta_{\text{Imp}}}=0)-\mathbb{P}(y_{\theta_s-\frac{\beta}{\alpha}}=0))\\
&\hspace{14em}(\because \forall x\in[\theta_s-\frac{\beta}{\alpha},\theta_{\text{Imp}}-\frac{\beta}{\alpha}],\quad -\mathbb{P}(y_x=0)\geq-\mathbb{P}(y_{\theta_s-\frac{\beta}{\alpha}}=0))\\
&> (F(\theta_{\text{Imp}}-\frac{\beta}{\alpha})-F(\theta_s-\frac{\beta}{\alpha}))(1-2\mathbb{P}(y_{\theta^\star}=0))\\
&\hspace{16em}(\because -\mathbb{P}(y_{\theta_{\text{Imp}}}=0)>-\mathbb{P}(y_{\theta_s-\frac{\beta}{\alpha}}=0)=-\mathbb{P}(y_{\theta^\star}=0))\\
&=0 \hspace{28em}(\because \mathbb{P}(y_{\theta^\star}=0)=\tfrac{1}{2})
\end{aligned}
$$

□

Case 1: Lower bound ($\theta_{\text{Imp}}\geq\theta^\star$) Now, let us assume that $\theta_{\text{Imp}}<\theta^\star$.

$$
\begin{aligned}
\texttt{err}_{\text{Imp}}(f_{\theta_{\text{Imp}}}) &= \mathbb{E}[\mathbb{1}(f_{\theta_{\text{Imp}}}(x')\neq y_{x'})]\\
&= \mathbb{E}[\mathbb{1}(f_{\theta_{\text{Imp}}}(x')\neq y_{x'},x\in R_{\theta_{\text{Imp}}})]+\mathbb{E}[\mathbb{1}(f_{\theta_{\text{Imp}}}(x')\neq y_{x'},x\notin R_{\theta_{\text{Imp}}})]\\
&= \mathbb{P}(f_{\theta_{\text{Imp}}}(x')\neq y_{x'},x\in R_{\theta_{\text{Imp}}})+\mathbb{P}(f_{\theta_{\text{Imp}}}(x')\neq y_{x'},x\notin R_{\theta_{\text{Imp}}})
\end{aligned}
$$

$R_{\theta_{\text{Imp}}}:=[\theta_{\text{Imp}}-\frac{\beta}{\alpha},\theta_{\text{Imp}}]$ is the region such that agents in this region modify their features to get positive classification.

$$
\begin{aligned}
&= \mathbb{P}(y_{\theta_{\text{Imp}}}=0,x\in R_{\theta_{\text{Imp}}})+\mathbb{P}(f_{\theta_{\text{Imp}}}(x')\neq y_{x'},x\notin R_{\theta_{\text{Imp}}})\\
&= \mathbb{P}(y_{\theta_{\text{imp}}}=0,\,x\in R_{\theta_{\text{imp}}})+\mathbb{P}(y_x=0,\,x>\theta_{\text{Imp}})+\mathbb{P}(y_x=1,\,x<\theta_{\text{Imp}}-\frac{\beta}{\alpha})\\
&= \int_{\theta_{\text{imp}}-\frac{\beta}{\alpha}}^{\theta_{\text{imp}}}\mathbb{P}(y_{\theta_{\text{imp}}}=0)\,p_X(x)\,dx+\int_{\theta_{\text{imp}}}^{\infty}\mathbb{P}(y_x=0)\,p_X(x)\,dx\\
&\quad+\int_{-\infty}^{\theta_{\text{imp}}-\frac{\beta}{\alpha}}\mathbb{P}(y_x=1)\,p_X(x)\,dx\\
&= \texttt{err}_{\text{Imp}}(f_{\theta^\star})+\mathbb{P}(y_{\theta_{\text{Imp}}}=0)(F(\theta_{\text{Imp}})-F(\theta_{\text{Imp}}-\frac{\beta}{\alpha}))
\end{aligned}
$$

$$
\begin{aligned}
&- \mathbb{P}(y_{\theta^\star} = 0)(F(\theta^\star) - F(\theta^\star - \frac{\beta}{\alpha})) + \int_{\theta_{\mathrm{Imp}}}^{\theta^\star} \mathbb{P}(y_x = 0)\, p_X(x)\, dx \\
&- \int_{\theta_{\mathrm{Imp}} - \frac{\beta}{\alpha}}^{\theta^\star - \frac{\beta}{\alpha}} \mathbb{P}(y_x = 1)\, p_X(x), dx \\
&> \mathtt{err}_{\mathrm{Imp}}(f_{\theta^\star}) \qquad \text{(Follows from the } \textit{Lemma C.2}\text{)}
\end{aligned}
$$

which contradicts improvement optimality condition of classifier, our assumption is false. Thus $\theta_{\mathrm{Imp}} \geq \theta^\star$

Case 2: Upper bound ($\theta_{\mathrm{Imp}} \leq \theta_s$) Now, let us assume that $\theta_{\mathrm{Imp}} > \theta_s$.

$$
\begin{aligned}
\mathtt{err}_{\mathrm{Imp}}(f_{\theta_{\mathrm{Imp}}}) &= \mathbb{E}[\mathbb{1}(f_{\theta_{\mathrm{Imp}}}(x') \neq y_{x'})] \\
&= \mathbb{P}(y_{\theta_{\mathrm{imp}}} = 0,\, x \in R_{\theta_{\mathrm{imp}}}\big) + \mathbb{P}(y_x = 0,\, x > \theta_{\mathrm{Imp}}) + \mathbb{P}(y_x = 1,\, x < \theta_{\mathrm{Imp}} - \frac{\beta}{\alpha}) \\
&= \int_{\theta_{\mathrm{imp}} - \frac{\beta}{\alpha}}^{\theta_{\mathrm{imp}}} \mathbb{P}(y_{\theta_{\mathrm{imp}}} = 0)\, p_X(x)\, dx + \int_{\theta_{\mathrm{imp}}}^{\infty} \mathbb{P}(y_x = 0)\, p_X(x)\, dx \\
&\quad + \int_{-\infty}^{\theta_{\mathrm{imp}} - \frac{\beta}{\alpha}} \mathbb{P}(y_x = 1)\, p_X(x)\, dx \\
&= \mathtt{err}_{\mathrm{Imp}}(f_{\theta_s}) + \mathbb{P}(y_{\theta_{\mathrm{Imp}}} = 0)(F(\theta_{\mathrm{Imp}}) - F(\theta_{\mathrm{Imp}} - \frac{\beta}{\alpha})) \\
&\quad - \mathbb{P}(y_{\theta_s} = 0)(F(\theta_s) - F(\theta_s - \frac{\beta}{\alpha})) - \int_{\theta_s}^{\theta_{\mathrm{Imp}}} \mathbb{P}(y_x = 0)\, p_X(x)\, dx \\
&\quad - \int_{\theta_s - \frac{\beta}{\alpha}}^{\theta_{\mathrm{Imp}} - \frac{\beta}{\alpha}} \mathbb{P}(y_x = 1)\, p_X(x), dx \\
&> \mathtt{err}_{\mathrm{Imp}}(f_{\theta_s}) \qquad \text{(Follows from the } \textit{Lemma} \text{ C.3)}
\end{aligned}
$$

which contradicts the improvement optimality condition, our assumption is false. Thus $\theta_{\mathrm{Imp}} \leq \theta_s$. □

D Additional Proofs from Section 4

Lemma D.1 (Uniform deviation bound for a VC class)**.** *Let H be a class of binary functions $h : \mathcal{Z} \to \{0, 1\}$ with $\mathrm{VC}(H) \leq d$. Let $Z_1, \ldots, Z_n \overset{\text{i.i.d.}}{\sim} P$, and define*

$$\Gamma(S) := \sup_{h \in H} \left| \mathbb{E}[h(Z)] - \frac{1}{n} \sum_{i=1}^{n} h(Z_i) \right|.$$

Then

$$\mathbb{E}_S[\Gamma(S)] \leq \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d \log(en/d)}{n}}.$$

Moreover, with probability at least $1 - \delta$,

$$\Gamma(S) \leq \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d \log(en/d)}{n}} + \sqrt{\frac{2 \log(1/\delta)}{n}}.$$

Proof. Let $S' = \{Z'_1, \ldots, Z'_n\}$ be an independent ghost sample drawn from P, and let $\sigma_1, \ldots, \sigma_n$ be independent Rademacher random variables. By symmetrization,

$$\mathbb{E}_S[\Gamma(S)] \leq 2\mathbb{E}_{S,\sigma} \left[\sup_{h \in H} \left| \frac{1}{n} \sum_{i=1}^{n} \sigma_i h(Z_i) \right| \right] = 2\mathbb{E}_S[\widehat{\mathfrak{R}}_S(H)].$$

Since H is binary and has VC dimension at most d, Sauer's lemma gives

$$\Pi_H(n) \le \left(\frac{en}{d}\right)^d,$$

where $\Pi_H(n)$ is the growth function of H. Combining this with the standard Rademacher bound for binary VC classes yields

$$\mathbb{E}_S[\widehat{\mathfrak{R}}_S(H)] \le \frac{1}{\sqrt{n}} + 2\sqrt{\frac{2d\log(en/d)}{n}}.$$

Therefore,

$$\mathbb{E}_S[\Gamma(S)] \le 2\left(\frac{1}{\sqrt{n}} + 2\sqrt{\frac{2d\log(en/d)}{n}}\right) = \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d\log(en/d)}{n}}.$$

This proves the expectation bound.

For the high-probability part, observe that changing one sample point Z_i changes $\frac{1}{n}\sum_{i=1}^n h(Z_i)$ by at most $1/n$ for every $h \in H$, and therefore changes $\Gamma(S)$ by at most $1/n$. Hence $\Gamma(S)$ satisfies the bounded-difference condition. By McDiarmid's inequality, with probability at least $1-\delta$,

$$\Gamma(S) \le \mathbb{E}_S[\Gamma(S)] + \sqrt{\frac{2\log(1/\delta)}{n}}.$$

Substituting the expectation bound above proves the claimed high-probability inequality. $\square$

Lemma D.2 (Bounded difference for uniform deviations)**.** *Let H be a class of functions $h : \mathcal{Z} \to [0,1]$, and let*

$$\Gamma(S) := \sup_{h\in H}\left|\mathbb{E}[h(Z)] - \frac{1}{n}\sum_{i=1}^n h(z_i)\right|,$$

where $S = (z_1, \dots, z_n)$. If S and S' differ in exactly one coordinate, then

$$|\Gamma(S) - \Gamma(S')| \le \frac{1}{n}.$$

Proof. Suppose S and S' differ only in the j-th coordinate. For every $h \in H$,

$$\left|\frac{1}{n}\sum_{i=1}^n h(z_i) - \frac{1}{n}\sum_{i=1}^n h(z_i')\right| = \frac{1}{n}|h(z_j) - h(z_j')| \le \frac{1}{n},$$

because $h \in [0,1]$. Taking the supremum over $h \in H$ preserves the same bound. $\square$

Theorem 4.2. *Fix a measurable tie-breaking rule so that x^f is well defined for every $x \in \mathcal{X}_0$ and $f \in \mathcal{F}$. Let $x_1, \dots, x_n \overset{\text{i.i.d.}}{\sim} \mathcal{D}$, draw independent $U_1, \dots, U_n \sim \mathrm{Unif}[0,1]$, and define the oracle label for classifier f by $y_{x_i^f} := \mathbf{1}\{U_i \le \eta(x_i^f)\}$; equivalently, for every fixed f, $y_{x_i^f} \sim \mathrm{Bernoulli}(\eta(x_i^f))$. Define $\widehat{\mathit{err}}_{\mathrm{Imp}}(f) := \frac{1}{n}\sum_{i=1}^n \mathbf{1}[f(x_i^f) \ne y_{x_i^f}]$. From Lemma 4.1 , induced oracle-loss class $H := \{(x,u) \mapsto \mathbf{1}[f(x^f) \ne \mathbf{1}\{u \le \eta(x^f)\}] : f \in \mathcal{F}\}$ has finite VC dimension d_Δ. Then, for every $\delta > 0$, with probability at least $1-\delta$ over the draw of $(x_i, U_i)_{i=1}^n$,*

$$\sup_{f\in\mathcal{F}} |\mathit{err}_{\mathrm{Imp}}(f) - \widehat{\mathit{err}}_{\mathrm{Imp}}(f)| \le \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d_\Delta\log(en/d_\Delta)}{n}} + \sqrt{\frac{2\log(1/\delta)}{n}}.$$

Proof. For every $f \in \mathcal{F}$, define the induced oracle-loss function $h_f : \mathcal{X}_0 \times [0,1] \to \{0,1\}$ by

$$h_f(x,u) := \mathbf{1}\{f(x^f) \ne \mathbf{1}[u \le \eta(x^f)]\}.$$

The measurable tie-breaking rule ensures that x^f is well defined, and hence h_f is a well-defined binary function. Let $Z = (X,U)$, where $X \sim \mathcal{D}$ and $U \sim \mathrm{Unif}[0,1]$ independently, and let $Z_i = (x_i, U_i)$

for $i \in [n]$. Since $U \sim \mathrm{Unif}[0,1]$, for every fixed f, the random variable $\mathbf{1}[U \le \eta(X^f)]$ is distributed as $\mathrm{Bernoulli}(\eta(X^f))$. Therefore,

$$\mathbb{E}_Z[h_f(Z)] = \mathbb{E}_{X,U}\big[\mathbf{1}\{f(X^f) \neq \mathbf{1}[U \le \eta(X^f)]\}\big] = \mathtt{err}_{\mathrm{Imp}}(f).$$

Moreover, by the oracle-label construction $y_{x_i^f} := \mathbf{1}\{U_i \le \eta(x_i^f)\}$, we have

$$\frac{1}{n}\sum_{i=1}^{n} h_f(Z_i) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}[f(x_i^f) \neq y_{x_i^f}] = \widehat{\mathtt{err}}_{\mathrm{Imp}}(f).$$

Hence the uniform generalization gap in the theorem is exactly

$$\Gamma(S) := \sup_{f\in\mathcal{F}} \left| \mathbb{E}_Z[h_f(Z)] - \frac{1}{n}\sum_{i=1}^{n} h_f(Z_i) \right|,$$

where $S = (Z_1, \dots, Z_n)$. By assumption, the induced oracle-loss class $H = \{h_f : f \in \mathcal{F}\}$ has VC dimension at most d_Δ. Applying Lemma D.1 to this class gives

$$\mathbb{E}_S[\Gamma(S)] \le \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d_\Delta \log(en/d_\Delta)}{n}}.$$

Next, by Lemma D.2, changing one augmented sample point $Z_i = (x_i, U_i)$ changes $\Gamma(S)$ by at most $1/n$. Therefore, McDiarmid's inequality implies that, with probability at least $1-\delta$,

$$\Gamma(S) \le \mathbb{E}_S[\Gamma(S)] + \sqrt{\frac{2\log(1/\delta)}{n}}.$$

Combining the last two displays yields

$$\sup_{f\in\mathcal{F}} |\mathtt{err}_{\mathrm{Imp}}(f) - \widehat{\mathtt{err}}_{\mathrm{Imp}}(f)| \le \frac{2}{\sqrt{n}} + 4\sqrt{\frac{2d_\Delta \log(en/d_\Delta)}{n}} + \sqrt{\frac{2\log(1/\delta)}{n}}.$$

This proves the theorem. $\square$

E Generalization Bounds

In this section, we prove Theorem 4.4. The analysis is for the plug-in version of STRAT-IMP-AWARE: the unknown class-probability function η is replaced by an independently estimated plug-in estimator $\widehat{\eta}$, and post-response labels are generated according to $\widehat{\eta}$ at the selected best response. Throughout this section, let $r := \sup_{x\in\mathcal{X}_0} \|x\|_2 < \infty$, $\rho_{\mathrm{Imp}} := r + \beta/\min_{j\in[d]} \alpha_j$, $\varepsilon_\eta := \sup_{z\in\mathcal{A}} |\widehat{\eta}(z) - \eta(z)|$, and $\mathcal{W}_{k,B} := \{(w,b) : w \in \mathbb{R}^d_{\ge 0}, \|w\|_2 \le k,\ |b| \le B\}$. For $(w,b) \in \mathcal{W}_{k,B}$, define $S(w) := \beta \max_{j\in[d]} w_j/\alpha_j$, $M_{w,b}(x) := w^\top x - b + S(w)$, and $C_{k,B} := k\rho_{\mathrm{Imp}} + B$. Since $x \in \mathcal{X}_0$, $\|w\|_2 \le k$, and $|b| \le B$, we have $|M_{w,b}(x)| \le C_{k,B}$.

For $x \in \mathcal{X}_0$, let $x^{w,b} \in \mathcal{A}$ denote the selected best response to $f_{w,b}$. We define the plug-in strategic hinge loss by

$$\ell^{\widehat{\eta}}_{w,b}(x) := \widehat{\eta}(x^{w,b})(1 - M_{w,b}(x))_+ + (1 - \widehat{\eta}(x^{w,b}))(1 + M_{w,b}(x))_+.$$

Equivalently, $\ell^{\widehat{\eta}}_{w,b}(x)$ is the conditional expectation of the hinge loss when the post-response label is drawn as $Y^{\mathrm{Imp}} \mid x \sim \mathrm{Bernoulli}(\widehat{\eta}(x^{w,b}))$. Thus the sampling step in Algorithm 1 gives an unbiased stochastic implementation of this plug-in objective.

Define the population and empirical plug-in hinge risks as

$$L^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) := \mathbb{E}_{x\sim D}[\ell^{\widehat{\eta}}_{w,b}(x)], \qquad \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) := \frac{1}{n}\sum_{i=1}^{n} \ell^{\widehat{\eta}}_{w,b}(x_i).$$

Lemma E.1 (Uniform convergence for plug-in strategic hinge loss)**.** *Let $\widehat{\eta}$ be estimated independently of the training sample, and let $\mathcal{G}^{\widehat{\eta}}_{k,B} := \{\ell^{\widehat{\eta}}_{w,b} : (w,b) \in \mathcal{W}_{k,B}\}$. If $\widehat{\mathfrak{R}}_S(\mathcal{G}^{\widehat{\eta}}_{k,B}) \le C_{k,B}/\sqrt{n}$ and every loss in $\mathcal{G}^{\widehat{\eta}}_{k,B}$ is bounded by $1 + C_{k,B}$, then with probability at least $1-\delta$, uniformly over $(w,b) \in \mathcal{W}_{k,B}$, $L^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) \le \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) + 2C_{k,B}/\sqrt{n} + (1 + C_{k,B})\sqrt{2\log(2/\delta)/n}$.*

Proof. Condition on the independently estimated $\widehat{\eta}$. Then $\mathcal{G}^{\widehat{\eta}}_{k,B}$ is a fixed class of bounded losses over $x \sim D$. By the standard Rademacher generalization bound for bounded loss classes, with probability at least $1-\delta$, uniformly over $g \in \mathcal{G}^{\widehat{\eta}}_{k,B}$,

$$\mathbb{E}[g(X)] \leq \frac{1}{n}\sum_{i=1}^{n} g(x_i) + 2\widehat{\mathfrak{R}}_S(\mathcal{G}^{\widehat{\eta}}_{k,B}) + (1+C_{k,B})\sqrt{\frac{2\log(2/\delta)}{n}}.$$

Using $\widehat{\mathfrak{R}}_S(\mathcal{G}^{\widehat{\eta}}_{k,B}) \leq C_{k,B}/\sqrt{n}$ gives the claim. $\square$

Theorem 4.4. *Suppose Assumptions 1 and 2 hold, $\mathcal{X}_0$ is bounded with $r := \sup_{x\in\mathcal{X}_0}\|x\|_2 < \infty$, and $\widehat{\eta}$ is estimated independently of the training sample. Assume* STRAT-IMP-AWARE *uses the plug-in oracle $\widehat{\eta}$, a fixed measurable best-response rule, and returns $(\widehat{w},\widehat{b})$ with $\|\widehat{w}\|_2 \leq k$ and $|\widehat{b}| \leq B$. Then, for every $\delta \in (0,1)$, with probability at least $1-\delta$,*

$$\mathtt{err}_{\mathrm{Imp}}(f_{\widehat{w},\widehat{b}}) \leq \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(\widehat{w},\widehat{b}) + \frac{2(k\rho_{\mathrm{Imp}}+B)}{\sqrt{n}} + (1+k\rho_{\mathrm{Imp}}+B)\sqrt{\frac{2\log(2/\delta)}{n}} + 2(k\rho_{\mathrm{Imp}}+B)\varepsilon_\eta.$$

Proof of Theorem 4.4. Let $\mathcal{W}_{k,B} := \{(w,b) : w \in \mathbb{R}^d_{\geq 0}, \|w\|_2 \leq k, |b| \leq B\}$ and set $C_{k,B} := k\rho_{\mathrm{Imp}} + B$. Fix $(w,b) \in \mathcal{W}_{k,B}$, and let $x^{w,b} \in \mathcal{A}$ be the selected best response of $x \in \mathcal{X}_0$ to $f_{w,b}$. Write $S(w) := \beta \max_{j\in[d]} w_j/\alpha_j$ and $M_{w,b}(x) := w^\top x - b + S(w)$. By the strategic-shift characterization under decomposable costs, the post-response prediction satisfies $f_{w,b}(x^{w,b}) = \mathbb{1}\{M_{w,b}(x) \geq 0\}$.

The conditional improvement-aware 0/1 loss at x is $\eta(x^{w,b})\mathbb{1}\{M_{w,b}(x) < 0\} + (1-\eta(x^{w,b}))\mathbb{1}\{M_{w,b}(x) \geq 0\}$. Since $\mathbb{1}\{M < 0\} \leq (1-M)_+$ and $\mathbb{1}\{M \geq 0\} \leq (1+M)_+$, we get

$$\mathtt{err}_{\mathrm{Imp}}(f_{w,b}) \leq L^{\eta}_{\mathrm{hinge}}(w,b),$$

where

$$L^{\eta}_{\mathrm{hinge}}(w,b) := \mathbb{E}_{x\sim D}\left[\eta(x^{w,b})(1-M_{w,b}(x))_+ + (1-\eta(x^{w,b}))(1+M_{w,b}(x))_+\right].$$

Next define the plug-in hinge risk

$$L^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) := \mathbb{E}_{x\sim D}\left[\widehat{\eta}(x^{w,b})(1-M_{w,b}(x))_+ + (1-\widehat{\eta}(x^{w,b}))(1+M_{w,b}(x))_+\right].$$

Then

$$\left|L^{\eta}_{\mathrm{hinge}}(w,b) - L^{\widehat{\eta}}_{\mathrm{hinge}}(w,b)\right| \leq \mathbb{E}_{x\sim D}\left[|\eta(x^{w,b}) - \widehat{\eta}(x^{w,b})|\,|(1-M_{w,b}(x))_+ - (1+M_{w,b}(x))_+|\right].$$

Since $x^{w,b} \in \mathcal{A}$, we have $|\eta(x^{w,b}) - \widehat{\eta}(x^{w,b})| \leq \varepsilon_\eta$. Moreover, $|(1-m)_+ - (1+m)_+| \leq 2|m|$ for all $m \in \mathbb{R}$. Finally, for all $x \in \mathcal{X}_0$,

$$|M_{w,b}(x)| \leq |w^\top x| + |b| + S(w) \leq kr + B + \frac{\beta k}{\min_{j\in[d]}\alpha_j} = k\rho_{\mathrm{Imp}} + B = C_{k,B}.$$

Hence

$$L^{\eta}_{\mathrm{hinge}}(w,b) \leq L^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) + 2C_{k,B}\varepsilon_\eta.$$

By Lemma E.1, with probability at least $1-\delta$, uniformly over $(w,b) \in \mathcal{W}_{k,B}$,

$$L^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) \leq \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) + \frac{2C_{k,B}}{\sqrt{n}} + (1+C_{k,B})\sqrt{\frac{2\log(2/\delta)}{n}}.$$

Combining the last three displays gives, uniformly over $(w,b) \in \mathcal{W}_{k,B}$,

$$\mathtt{err}_{\mathrm{Imp}}(f_{w,b}) \leq \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(w,b) + \frac{2C_{k,B}}{\sqrt{n}} + (1+C_{k,B})\sqrt{\frac{2\log(2/\delta)}{n}} + 2C_{k,B}\varepsilon_\eta.$$

Since STRAT-IMP-AWARE returns $(\widehat{w},\widehat{b}) \in \mathcal{W}_{k,B}$, substituting $(w,b) = (\widehat{w},\widehat{b})$ and expanding $C_{k,B} = k\rho_{\mathrm{Imp}} + B$ proves the theorem. $\square$

F Oracle Free Extension

We note that our experimental implementation already operates without oracle access: Algorithm 1 evaluates a calibrated logistic regression estimator $\widehat{\eta}$ at each agent's best response $\mathrm{BR}(x; w, b)$ and samples the post-response label as $y_{\mathrm{Imp}} \sim \mathrm{Bernoulli}(\widehat{\eta}(\mathrm{BR}(x; w, b)))$, rather than querying a true post-deployment label oracle. The two stages use distinct objectives by construction: $\widehat{\eta}$ is fit by cross-entropy minimization, since it is a proper scoring rule yielding calibrated probabilities, and is then evaluated at the best response to sample post-response labels in Algorithm 1, while the classifier is trained by the empirical plug-in strategic hinge loss minimized by STRAT-IMP-AWARE of Theorem 4.4. Our experiments therefore validate the oracle-free regime, though the original theoretical analysis (Theorem 4.4) assumes exact oracle labels. We now formalize this by replacing the oracle with access to a class-probability estimator $\widehat{\eta} : \mathcal{X} \to [0, 1]$, learned from pre-deployment data. The modified algorithm is identical to Algorithm 3 except that the ORACLE subroutine (Algorithm 1) computes the post-response label using $\widehat{\eta}$ in place of η. This reflects the practical implementation and requires no additional assumptions beyond the availability of a consistent estimator $\widehat{\eta}$. The classification-stage guarantee depends on $\widehat{\eta}$ only through $\varepsilon_\eta = \sup_{z\in\mathcal{A}} |\widehat{\eta}(z) - \eta(z)|$, and not on the loss used to obtain it; cross-entropy is therefore one sufficient choice, and any independently estimated $\widehat{\eta}$ with $\varepsilon_\eta = \mathcal{O}(m^{-1/2})$ yields the same conclusion.

Theorem F.1 (Oracle-free guarantee)**.** *Suppose the assumptions of Theorem 4.4 hold and let $\|\widehat{w}\|_2 \le B$. Further, let the estimator $\widehat{\eta}(x) = g(\widehat{w}^\top x)$ be obtained by cross entropy (logistic) minimization on an independent sample $S_\eta = \{(X_i, Y_i)\}_{i=1}^m$ where $Y_i \mid X_i \sim \mathrm{Bernoulli}(\eta(X_i))$. and let $\ell(w; (x, y))$ denote the cross entropy loss with empirical loss $\widehat{L}_m(w) = \frac{1}{m}\sum_{i=1}^m \ell(w; (X_i, Y_i))$. Furthermore, assume that the feature space is bounded i.e., $\|x\|_2 \le r$ for all $x \in \mathcal{X}$. and there exists $\tau \in (0, 1/2)$ such that $\tau \le g(w^\top x) \le 1 - \tau$ for all $x \in \mathcal{X},\ w \in \mathcal{W} \subset \mathbb{R}^d$ with a convex set $\mathcal{W}$; let g be L_g Lipschitz. Let $\widehat{w} \in \arg\min_{w\in\mathcal{W}} \widehat{L}_m(w)$ and with probability at least $1 - \delta/4$, the empirical risk $\widehat{L}_m$ is κ-strongly convex on $\mathcal{W}$. Then, with probability at least $1 - \delta$,*

$$\mathtt{err}_{\mathrm{Imp}}(f_{\widehat{w},\widehat{b}}) \le \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(\widehat{w}, \widehat{b}) + (1+k\rho_{\mathrm{Imp}}+B)\sqrt{\frac{2\log(2/\delta)}{n}} + 2(k\rho_{\mathrm{Imp}}+B)\left(\frac{1}{\sqrt{n}} + \frac{2L_g^2 r^2}{\kappa\tau(1-\tau)}\sqrt{\frac{2(d+\log(4/\delta))}{m}}\right)$$

In particular, if d, r, L_g, κ^{-1}, and τ^{-1} are fixed constants, then the additional oracle-free estimation price is $\mathcal{O}(m^{-1/2})$, and the overall rate is $\mathcal{O}(n^{-1/2} + m^{-1/2})$.

Proof. By Theorem 4.4, applied with failure probability δ, we have with probability at least $1 - \delta$,

$$\mathtt{err}_{\mathrm{Imp}}(f_{\widehat{w},\widehat{b}}) \le \widehat{L}^{\widehat{\eta}}_{\mathrm{hinge}}(\widehat{w}, \widehat{b}) + \frac{2(k\rho_{\mathrm{Imp}} + B)}{\sqrt{n}} + (1+k\rho_{\mathrm{Imp}}+B)\sqrt{\frac{2\log(2/\delta)}{n}} + 2(k\rho_{\mathrm{Imp}}+B)\varepsilon_\eta.$$

Using the assumed norm bound $\|\widehat{w}\|_2 \le B$, it remains to control

$$\sup_{z\in\mathcal{X}} |\widehat{\eta}(z) - \eta(z)|.$$

Let $w^\star$ denote the true single-index parameter, so that $\eta(x) = g(w^{\star\top} x)$ and $(Y|X = x) \sim \mathrm{Bernoulli}(g(w^{\star\top} x))$; equivalently, $w^\star$ is the population minimizer of the cross-entropy risk $\mathbb{E}[\ell(w; (x, y))]$. We assume $w^\star \in \mathcal{W}$. The estimator $\widehat{w}$ minimizes the emperical risk L_m over $\mathcal{W}$. We first prove a high-probability bound on $\|\widehat{w} - w^\star\|_2$. Define

$$\mu_w(x) := g(w^\top x).$$

The gradient of the loss is

$$\nabla_w \ell(w; (x, y)) = \frac{g'(w^\top x)}{\mu_w(x)(1 - \mu_w(x))}\big(\mu_w(x) - y\big)x.$$

At the true parameter $w^\star$, since

$$Y \mid X = x \sim \mathrm{Bernoulli}(\mu_{w^\star}(x)),$$

we have

$$\mathbb{E}[\nabla_w \ell(w^\star; (X, Y)) \mid X] = 0.$$

Thus

$$\mathbb{E}[\nabla \widehat{L}_m(w^\star)] = 0.$$

Moreover, by the boundedness assumptions,

$$\|\nabla_w \ell(w^\star; (X, Y))\|_2 \le \frac{L_g}{\tau(1-\tau)} \|X\|_2 \le \frac{L_g r}{\tau(1-\tau)}.$$

Let

$$M := \frac{L_g r}{\tau(1-\tau)}.$$

By a standard vector Hoeffding inequality for bounded mean-zero random vectors, with probability at least $1 - \delta/4$,

$$\|\nabla \widehat{L}_m(w^\star)\|_2 \le M \sqrt{\frac{2(d + \log(4/\delta))}{m}}.$$

On the event that $\widehat{L}_m$ is κ-strongly convex on $\mathcal{W}$, we have, for every $w \in \mathcal{W}$,

$$\widehat{L}_m(w) \ge \widehat{L}_m(w^\star) + \langle \nabla \widehat{L}_m(w^\star), w - w^\star \rangle + \frac{\kappa}{2} \|w - w^\star\|_2^2.$$

Taking $w = \widehat{w}$, and using the empirical optimality of $\widehat{w}$, namely

$$\widehat{L}_m(\widehat{w}) \le \widehat{L}_m(w^\star),$$

we get

$$0 \ge \langle \nabla \widehat{L}_m(w^\star), \widehat{w} - w^\star \rangle + \frac{\kappa}{2} \|\widehat{w} - w^\star\|_2^2.$$

Therefore,

$$\frac{\kappa}{2} \|\widehat{w} - w^\star\|_2^2 \le -\langle \nabla \widehat{L}_m(w^\star), \widehat{w} - w^\star \rangle.$$

By Cauchy–Schwarz,

$$\frac{\kappa}{2} \|\widehat{w} - w^\star\|_2^2 \le \|\nabla \widehat{L}_m(w^\star)\|_2 \|\widehat{w} - w^\star\|_2.$$

If $\widehat{w} \neq w^\star$, dividing by $\|\widehat{w} - w^\star\|_2$ gives

$$\|\widehat{w} - w^\star\|_2 \le \frac{2}{\kappa} \|\nabla \widehat{L}_m(w^\star)\|_2.$$

The same inequality is trivially true when $\widehat{w} = w^\star$. Hence, with probability at least $1 - \delta/4$,

$$\|\widehat{w} - w^\star\|_2 \le \frac{2M}{\kappa} \sqrt{\frac{2(d + \log(4/\delta))}{m}}.$$

We now translate this parameter-estimation bound into a uniform class-probability estimation bound. For any $z \in \mathcal{X}$,

$$|\widehat{\eta}(z) - \eta(z)| = |g(\widehat{w}^\top z) - g(w^{\star\top} z)|.$$

Since $g(\cdot)$ is L_g-Lipschitz,

$$|\widehat{\eta}(z) - \eta(z)| \le L_g |(\widehat{w} - w^\star)^\top z|.$$

By Cauchy–Schwarz and $\|z\|_2 \le r$,

$$|\widehat{\eta}(z) - \eta(z)| \le L_g r \|\widehat{w} - w^\star\|_2.$$

Taking the supremum over $z \in \mathcal{X}$, we obtain

$$\sup_{z \in \mathcal{X}} |\widehat{\eta}(z) - \eta(z)| \le L_g r \|\widehat{w} - w^\star\|_2.$$

Substituting the bound on $\|\widehat{w} - w^\star\|_2$ gives

$$\sup_{z \in \mathcal{X}} |\widehat{\eta}(z) - \eta(z)| \le \frac{2 L_g r M}{\kappa} \sqrt{\frac{2(d + \log(4/\delta))}{m}}.$$

Using

$$M = \frac{L_g r}{\tau(1-\tau)},$$

we get

$$\sup_{z\in\mathcal{X}} |\widehat{\eta}(z) - \eta(z)| \leq \frac{2L_g^2 r^2}{\kappa\tau(1-\tau)}\sqrt{\frac{2(d+\log(4/\delta))}{m}}.$$

Finally, combining this event with the event from Theorem 4.4 by a union bound yields the claimed probability $1-\delta$ and the stated oracle-free generalization bound. □

Remark F.2. The bound involves two sample sizes: n, the size of the classifier's training sample entering through Theorem 4.4, and $m = |S_\eta|$, the size of the independent sample used to fit $\widehat{\eta}$. The first two terms decay as $\mathcal{O}(n^{-1/2})$, and the estimation terms as $\mathcal{O}(m^{-1/2})$. In particular, when $m \gtrsim n$ the estimation term is dominated by the classifier term and the overall oracle-free rate is $\mathcal{O}(n^{-1/2})$.

G Experiments: Supplementary Details

In all experiments, we set the random seed to 42 to ensure reproducibility and consistency across all runs. For all experiments, each dataset is split into training (70%) and test (30%) sets. During training, we map the labels to $\{-1, 1\}$ to calculate the margin-based loss. This ensures the classifier learns a clear decision boundary that maximizes the distance between classes, providing a standard baseline for non-strategic settings. The classifier is trained using an objective that anticipates agent manipulation. Specifically, the model identifies the feature with the highest return on investment calculated as the ratio of the model weight to the manipulation cost (w_k/α_k). During training, we adjust the loss by adding a strategic gain term, which represents the maximum score change an agent can achieve within their budget β. This ensures the decision boundary is optimized for an environment where agents shift features to flip their predicted labels.

Compute Resources. All the models are trained using the specifications, CPU - Intel(R) Xeon(R) Gold 5318Y CPU @ 2.10GHz, GPU - NVIDIA RTX 6000

G.1 Validation of Modelling Assumptions

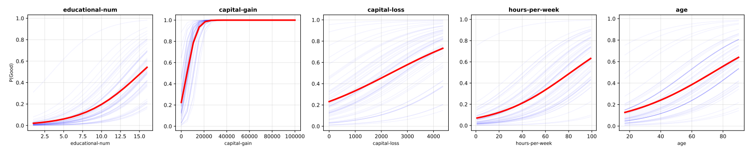
In the real-world deployment of machine learning models, user has different types of attributes (Numerical, categorical, etc.), among which only selected features are incentivized to manipulate; understanding these features is essential to quantify the user's improvement. In the following, we provide an explanation of these features for the datasets.A feature can be numerical (real-valued) or categorical (discrete finite set). In real-world settings, numerical features are often actionable; however, a user will typically manipulate a specific numerical feature if the model incentivizes them to do so to alter their predicted outcome. Furthermore all the categorical features can be classified as nominal and ordinal features. We observe that nominal features (e.g., race, gender) are typically immutable and therefore non-manipulable in this context. We observe for the non-negative weights of the optimal linear classifier, the relationship between $\eta(x)$ and the features in Figure 4 for the Adult, Heloc, and law school , and ACS Income datasets respectively, following the Assumption 1. We consider these features while training the classifiers.

G.2 Synthetic Dataset

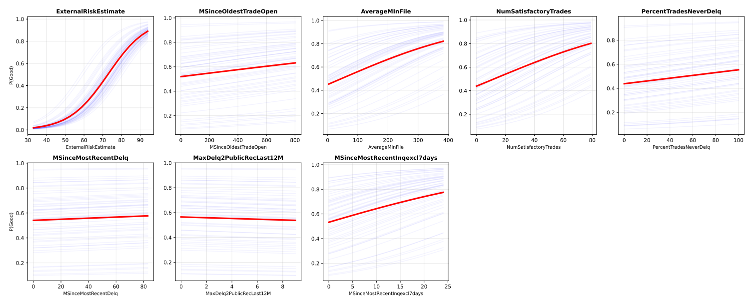
We generate a synthetic dataset under a fully specified probabilistic model to provide a controlled environment for evaluating strategic classification methods. The dataset consists of $n = 20{,}000$ samples in $d = 8$ dimensions, where each feature vector $x \in \mathbb{R}^d$ is independently drawn from a multivariate normal distribution with zero mean and covariance matrix $\Sigma = 100I_d$. This ensures that features are uncorrelated while maintaining sufficiently high variance.

The label-generating process follows a generalized linear model. We fix a weight vector $w = \mathbf{1} \in \mathbb{R}^d$, and define the conditional probability of the positive class as

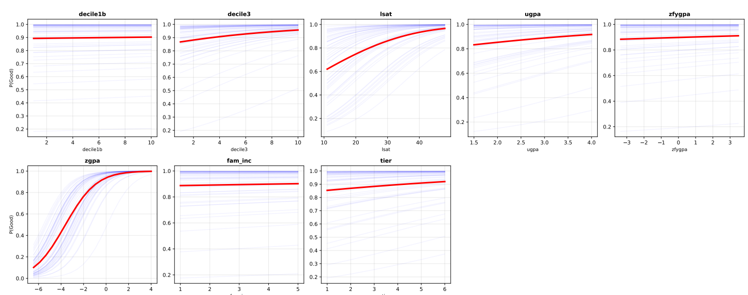
$$\eta(x) = \sigma(w^\top x),$$



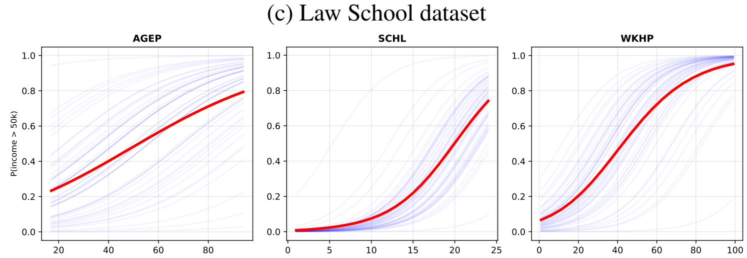
(a) Adult dataset



(b) HELOC dataset



(c) Law School dataset



(d) ACSIncome dataset

Figure 4: Improvable features across the datasets.

where $\sigma(\cdot)$ denotes the logistic sigmoid function. Labels are then sampled according to

$$y \sim \text{Bernoulli}(\eta(x)),$$

introducing stochasticity consistent with a well-specified logistic model.

G.3 Comparision with Benchmark Algorithms

We consider two Benchmark Algorithms one for Strategic Classifier from Levanon and Rosenfeld [27] and an Improvement aware classifier from Attias et al. [5] and use them for performance analysis of our Improvement aware classifier STRAT-IMP-AWARE. The details of the implementations as follows.

SERM [27]: We implement a strategic learning baseline that explicitly accounts for feature manipulation under the decomposable cost framework. Let the classifier be parameterized by a weight vector w, and given below the decomposable manipulation cost

$$c(x, x') = \sum_{j=1}^{d} \alpha_j (x'_j - x_j)^+,$$

where $\alpha_j > 0$ denotes the manipulation cost associated with feature j. we modify the algorithm of

The classifier outputs a linear score

$$f(x) = w^\top x.$$

Under this cost structure, the optimal strategic shift admits the closed-form expression

$$S(w) = \beta \max_{j \in [d]} \frac{w_j}{\alpha_j},$$

where $\beta > 0$ represents the utility gained from receiving a positive classification.

During training, labels are transformed from $\{0, 1\}$ to $\{-1, +1\}$ using

$$\tilde{y} = 2y - 1.$$

The strategic margin for a sample x_i is computed as

$$\tilde{y}_i \left(f(x_i) + S(w) \right).$$

The model is trained by minimizing the strategic hinge loss

$$\mathcal{L}(w) = \frac{1}{n} \sum_{i=1}^{n} \max\left(0,\, 1 - \tilde{y}_i \left(f(x_i) + S(w) \right)\right).$$

Thus, the classifier is optimized directly against the strategically manipulated decision environment induced by the deployed model. Optimization is performed using mini-batch stochastic gradient descent. At each iteration, the classifier predictions are computed, the strategic shift $S(w)$ is evaluated from the current model parameters, the strategic hinge loss is calculated, and gradients are backpropagated to update the classifier parameters.

Attias et al. [5] We implement a baseline for improvement aware classifier following the framework of PAC Learning with Improvements [5]. The method models strategic agents as individuals who iteratively improve their feature vectors in order to obtain a positive classification from the deployed model.

A decision-tree classifier f^* is first trained to represent the ground-truth qualification function. Specifically, a Classification and Regression Tree (CART) classifier with Gini impurity is used as the oracle labeler. The deployed classifier h is then trained on labels generated by f^*.

The deployed model is parameterized as a two-layer neural network:

$$h(x) = \sigma(W_2 \phi(W_1 x + b_1) + b_2),$$

where $\phi(\cdot)$ denotes the ReLU activation function and $\sigma(\cdot)$ is the sigmoid output layer.

Training is performed using a weighted binary cross-entropy objective:

$$\mathcal{L}(h) = -\frac{1}{n} \sum_{i=1}^{n} \left[w_{\mathrm{FN}} y_i \log(\hat{y}_i) + w_{\mathrm{FP}} (1 - y_i) \log(1 - \hat{y}_i) \right],$$

where w_{FP} and w_{FN} control the relative penalties assigned to false positives and false negatives.

To model strategic improvement, agents iteratively modify their features using projected gradient descent. Given an original feature vector x, the agent solves

$$\min_{x' \in B_\infty(x, r)} \mathcal{L}(h(x'), 1),$$

where

$$B_\infty(x, r) = \{x' : \|x' - x\|_\infty \le r\}$$

denotes the feasible improvement region with budget r.

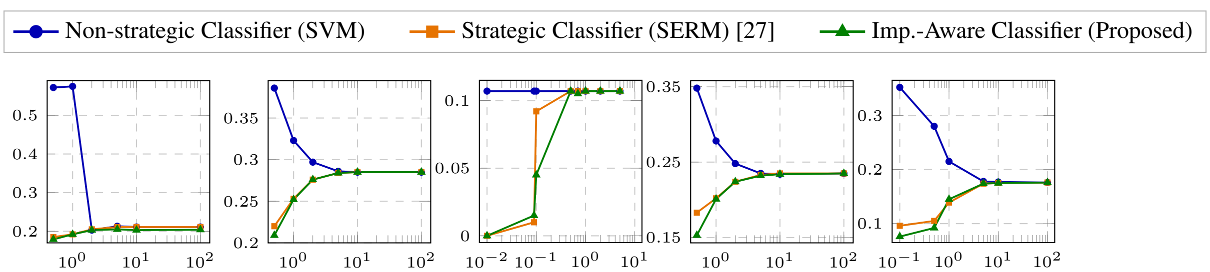
At each iteration, gradients are computed with respect to the objective encouraging positive classification, and the updated point is projected back into the feasible ℓ_∞-ball around the original feature vector. Only the predefined improvable features are allowed to change during this optimization process.

Experiments are conducted with budget value r = 1.0 and both standard evaluation and adversarial tie-breaking evaluation are reported. Under adversarial tie-breaking, if multiple manipulated representations achieve positive classification, preference is given to representations that remain negatively labeled by the oracle classifier f^*. Finally, using the model we evaluate the improvement rate metric and the values are tabulated in Table 2.

G.4 Ablation Studies

It is very important to understand how the classifiers perform with respect to the parameters, cost vector α and utility of positive classification β. To analyse the behaviour, we perform the following case studies with respect to the α and β. Throughout the study, the base cost vector α is initialized as an all-ones vector for all the datasets used and the manipulation budget is fixed at $\beta = 1$, for all the datasets.

Sensitivity analysis: Manipulation Cost (α). To analyze sensitivity, a uniform scaling factor is applied to α, and all three models are retrained for each scaled setting. Performance is measured using the improvement-aware error metric. Figure 5 shows how varying the manipulation cost multiplier affects this error. The theoretical ordering is consistently observed across domains: the improvement-aware classifier $f^\star_{\text{Imp}}$ achieves the lowest error, followed by the strategic classifier $f^\star_s$, and then the standard optimal linear classifier $f^\star$. As α becomes large, making manipulation increasingly costly, the performance of all three models converges, particularly on both real-world and synthetic datasets.



(a) Adult (b) HELOC (c) Law School (d) ACSIncome (e) Synthetic

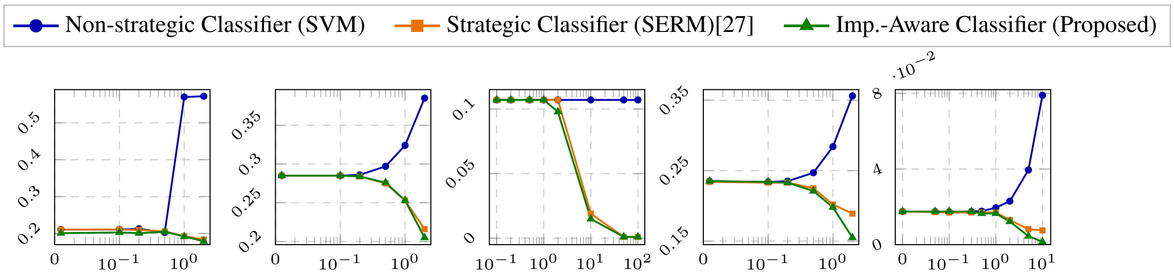
Figure 5: Effect of varying Alpha (α) hyperparameter on improvement error (`err_imp`) across the three classifiers for five distinct datasets.

Sensitivity analysis: Utility of Positive Classification (β). Figure 6 illustrates the improvement-aware strategic error across the Adult, HELOC, Law School, ACS Income datasets as the utility of the positive classification , β, is varied. The theoretical hierarchy is robustly satisfied. However, deviations from this expected trend emerge at the extremes of the β spectrum. When β is extremely low, the allowed budget for manipulation is negligible. Because agents essentially lack the capacity to manipulate their features, the strategic environment collapses back into a standard classification environment. As β becomes large, Incentivizing more users to manipulate, the performance of all three models diverges, particularly on both real-world and synthetic datasets.

H Improvement Probability Formulation

Let $\eta(x) = \mathbb{P}(y = 1|x)$ and $x, y \sim (\mathcal{D}, Ber(\eta(x)))$. We model improvement-aware strategic classification, such that the following condition holds -

$$\mathbb{P}(y^f = 1 \mid x) = \mathbb{P}(y = 1 \mid x^f)$$



(a) Adult (b) HELOC (c) Law School (d) ACSIncome (e) Synthetic

Figure 6: Effect of varying Beta (β) hyperparameter on improvement error (`err_imp`) across the three classifiers for four distinct Real world datasets and a Synthetic dataset

where y^f denotes the realized label after moving to point x^f through strategic manipulation, further:

$$\mathbb{P}(y^f = 1 \mid y = 1, x) = 1, \qquad \mathbb{P}(y^f = 1 \mid y = 0, x) = p$$

Now, by the law of total probability

$$\begin{aligned}
\mathbb{P}(y^f = 1|x) &= \mathbb{P}(y^f = 1, y = 1|x) + \mathbb{P}(y^f = 1, y = 0|x) \\
&= \mathbb{P}(y^f = 1 \mid y = 1, x)\mathbb{P}(y = 1 \mid x) + \mathbb{P}(y^f = 1 \mid y = 0, x)\mathbb{P}(y = 0 \mid x) \\
&= \eta(x) + p \cdot (1 - \eta(x))
\end{aligned}$$

From our modelling assumption, it follows that the improvement probability is $p = \frac{\eta(x^f) - \eta(x)}{1 - \eta(x)}$.

To implement the improvement-aware oracle model 1, we utilize p as the probability that an unqualified agent ($y_x = 0$) successfully flips their label, provided they strategically manipulated. The post-manipulation y_{x^f} is determined by a Bernoulli trial with parameter p. Formally:

$$y_{x^f} | (y_x = 0) \sim \texttt{Bernoulli}(p)$$

This ensures that label flipping is intrinsically tied to the strategic transition from $x \to x^f$, maintaining the probabilistic consistency of the improvement-aware framework.

I Additional Theoretical Results

Proposition I.1. *Consider a scoring function $h(x) = \sum_{i=1}^{d} \alpha_i \psi(x_i)$ with $\alpha \in \mathbb{R}^d_{>0}$. Let c_{sep} and c_{dec} denote separable and decomposable cost functions with respect to h and ψ, respectively. Let the feature transformation functions ψ_i are non-decreasing i.e. if $x'_i \geq x_i$ coordinate-wise then* i.e., *$\psi_i(x'_i) \geq \psi_i(x_i)$ for all $i \in [d]$. Then, for any $x, x' \in \mathcal{X}$, the costs satisfy $c_{sep}(x, x') = c_{dec}(x, x')$.*

Proof. Let $x, x' \in \mathcal{X}$. Define the feature-wise deviation $\delta_i = \psi(x'_i) - \psi(x_i)$.

Recall the definition of the decomposable cost:

$$c_{dec}(x, x') = \sum_{i=1}^{d} \alpha_i |\psi(x'_i) - \psi(x_i)|_+ = \sum_{i=1}^{d} \alpha_i \max(0, \delta_i)$$

Similarly, recall the definition of the separable cost, substituting the scoring function structure:

$$\begin{aligned}
c_{sep}(x, x') &= \max\left(0, \sum_{i=1}^{d} \alpha_i \psi(x'_i) - \sum_{i=1}^{d} \alpha_i \psi(x_i)\right) \\
&= \max\left(0, \sum_{i=1}^{d} \alpha_i \delta_i\right)
\end{aligned}$$

We now invoke the monotonicity condition, then $\psi(x_i') \geq \psi(x_i)$ for all $i \in [d]$. This implies $\delta_i \geq 0$ for all i.
Under this condition, we can clearly conclude that:

$$c_{dec}(x, x') = \sum_{i=1}^{d} \alpha_i \delta_i = c_{sep}(x, x')$$

$\square$